

Free Energies of Dilute Bose gases

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Abstract

We derive a upper bound on the free energy of a Bose gas system at density ϱ and temperature T . In combination with the lower bound derived previously by Seiringer [15], our result proves that in the low density limit, i.e., when $a^3\varrho \ll 1$, where a denotes the scattering length of the pair-interaction potential, the leading term of Δf the free energy difference per volume between interacting and ideal Bose gases is equal to $4\pi a(2\varrho^2 - [\varrho - \varrho_c]_+^2)$. Here, $\varrho_c(T)$ denotes the critical density for Bose-Einstein condensation (for the ideal gas), and $[\cdot]_+ = \max\{\cdot, 0\}$ denotes the positive part.

1 Introduction

The ground state energy, free energy are the fundamental properties of a quantum system and they have been intensively studied since the invention of the quantum mechanics. The recent progresses in experiments for the Bose-Einstein condensation, especially the achievement of Bose-Einstein condensation in dilute gases of alkali atoms in 1995 [1], have inspired re-examination of the theoretic foundation concerning the Bose system, e.g., [14], [12], [13], [7], [4], [17], [5] and [16] on ground state energy and [15] on free energy.

In the low density limit, the leading term of the ground state energy per volume was identified rigorously by Dyson (upper bound) [3] and Lieb-Yngvason (lower bound) [14] to be $4\pi a\varrho^2$, where a is the scattering length of the two-body potential and ϱ is the density. We note that $4\pi a\varrho^2$ is also the first leading term of ΔE the ground state energy difference per volume between interacting and ideal Bose gases. (The ground state energy per volume of ideal Bose gas is zero).

On the other hand, the first leading term of Δf the free energy difference between interacting and ideal Bose gases is the second leading order of the free energy. More specifically, when $a^3\varrho \ll 1$, where a denotes the scattering length of the pair-interaction potential, then

$$f(\varrho, T) = f_0(\varrho, T) + 4\pi a(2\varrho^2 - [\varrho - \varrho_c]_+^2) + o(a\varrho^2) \quad (1.1)$$

Here, f is the free energy per volume of interacting Bose gas, f_0 is the one of ideal Bose gas and $\varrho_c(T)$ denotes the critical density for Bose-Einstein condensation (for the ideal gas), and $[\cdot]_+ = \max\{\cdot, 0\}$ denotes the positive part. The lower bound on f has been proved in Seiringer's work [15]. In this paper, we prove the upper bound on f and obtain the main result (1.1)

The trial state we use in this proof is a new type, which was first used in [16]. Let ϕ_0 be the ground state of ideal Bose gas system. The trial state (pure state) Yau and Yin constructed for interacting Bose gases in [16] is almost equal to the following one

$$\begin{aligned} & \exp \left[\sum_k \sum_{v \sim \sqrt{\varrho}} 2\sqrt{\lambda_{k+v/2}\lambda_{-k+v/2}} a_{k+v/2}^\dagger a_{-k+v/2}^\dagger a_v a_0 \right. \\ & \left. + \sum_k c_k a_k^\dagger a_{-k}^\dagger a_0 a_0 \right] |\phi_0\rangle, \end{aligned} \quad (1.2)$$

(with suitably chosen c and λ). This trial state (pure state) in [16] is used to rigorously prove the upper bound of the second order correction to the

ground state energy, which was first computed by Lee-Yang [9] (see also Lee-Huang-Yang [8] and the recent paper by Yang [17] for results in other dimensions. Another derivation was later given by Lieb [10] using a self-consistent closure assumption for the hierarchy of correlation functions.)

We can rewrite the pure state (1.2) as follows

$$(1.2) = P_{(0,0)} P_{(0,\sqrt{\varrho})} |\phi_0\rangle \quad (1.3)$$

where

$$\begin{aligned} P_{(0,0)} &= \exp \left[\sum_k c_k a_k^\dagger a_{-k}^\dagger a_0 a_0 \right] \\ P_{(0,\sqrt{\varrho})} &= \exp \left[\sum_{k \sim 1} \sum_{v \sim \sqrt{\varrho}} 2\sqrt{\lambda_{k+v/2} \lambda_{-k+v/2}} a_{k+v/2}^\dagger a_{-k+v/2}^\dagger a_v a_0 \right] \end{aligned} \quad (1.4)$$

We note: $P_{(0,0)}$ represents the interactions between condensate and condensate, since two particles with momenta zero are annihilated ($a_0 a_0$) and two particles with high momentum are created ($a_k^\dagger a_{-k}^\dagger$). Similarly $P_{(0,\sqrt{\varrho})}$ represents the interaction between condensate and the particles with momentum $\sim O(\varrho^{1/2})$, since in this operator one particle with momentum zero and one with momentum $\sim O(\varrho^{1/2})$ are annihilated ($a_v a_0$) and other two particles with high momenta are created.

In this paper, we construct a trial state of the similar form. More specifically, let Γ_I be Gibbs state of ideal Bose gas at temperature T , the trial state we are going to use is very close to the following one

$$\Gamma \sim \left(P_{(\varrho^{1/3}, \varrho^{1/3})} P_{(0, \varrho^{1/3})} P_{(0,0)} \right) \Gamma_I \left(P_{(\varrho^{1/3}, \varrho^{1/3})} P_{(0, \varrho^{1/3})} P_{(0,0)} \right)^\dagger \quad (1.5)$$

where

$$\begin{aligned} P_{(0,0)} &= \exp \left[\sum_k c_k a_k^\dagger a_{-k}^\dagger a_0 a_0 \right] \\ P_{(0, \varrho^{1/3})} &= \exp \left[\sum_k \sum_{v \sim \varrho^{1/3}} 2\sqrt{\lambda_{k+v/2} \lambda_{-k+v/2}} a_{k+v/2}^\dagger a_{-k+v/2}^\dagger a_v a_0 \right] \\ P_{(\varrho^{1/3}, \varrho^{1/3})} &= \exp \left[\sum_k \sum_{u \neq v \sim \varrho^{1/3}} \sqrt{\lambda_{k+\frac{v+u}{2}} \lambda_{-k+\frac{v+u}{2}}} a_{k+\frac{v+u}{2}}^\dagger a_{-k+\frac{v+u}{2}}^\dagger a_v a_u \right] \end{aligned} \quad (1.6)$$

where constant 2 comes from the ordering of $a_v a_0$. As one can see $P_{(0,0)}$ represents the interactions between condensate and condensate, $P_{(0,\varrho^{1/3})}$ represents the interaction between condensate and the particles with momentum $\sim O(\varrho^{1/3})$ and $P_{(\varrho^{1/3},\varrho^{1/3})}$ represents the interaction between the particles with momentum $\sim O(\varrho^{1/3})$.

2 Model and Main results

2.1 Hamiltonian and Notations

We consider a Bose gas system which is composed of N same bosons and confined in a cubic box Λ of side L . The Hilbert space $\mathcal{H}_{N,\Lambda}$ for the system is the set of symmetric functions in $L^2(\Lambda^N)$. The Hamiltonian is given as

$$H_{N,\Lambda} = -\sum_{i=1}^N \Delta_i + \sum_{1 \leq i \neq j \leq N} V(|x_i - x_j|) \quad (2.1)$$

Here the two body interaction is given by a smooth, symmetric non-negative function $V(x)$ of fast decay. In particular, it has a finite scattering length, which we denote by a . As usually, we denote by $H_{N,\Lambda}^P$ ($H_{N,\Lambda}^D$) the Hamiltonians with periodic (Dirichlet) boundary conditions.

The dual space of Λ is $\Lambda^* := (\frac{2\pi}{L}\mathbb{Z})^3$. For a continuous function F on \mathbb{R}^3 , we have

$$\frac{1}{L^3} \sum_{p \in \Lambda^*} F(p) = \frac{1}{|\Lambda|} \sum_{p \in \Lambda^*} F(p) \xrightarrow{|\Lambda| \rightarrow \infty} \int_{\mathbb{R}^3} \frac{d^3 p}{(2\pi)^3} F(p)$$

The Fourier transform is defined as

$$\widehat{V}_p = \int_{\Lambda} e^{-ipx} V(x) dx, \quad V(x) = \frac{1}{|\Lambda|} \sum_{p \in \Lambda^*} e^{ipx} \widehat{V}_p$$

and then

$$\frac{1}{|\Lambda|} \sum_{p \in \Lambda^*} e^{ipx} = \delta_{\mathbb{R}^3}(x), \quad \int_{\Lambda} e^{ipx} dx = \delta_{\Lambda^*}(p)$$

where $\delta_{\mathbb{R}^3}(x)$ is the usual continuum delta function and the function $\delta_{\Lambda^*}(p) = |\Lambda| = L^d$ if $p = 0$ (otherwise it is zero) is the lattice delta-function. We will neglect the subscript, the argument indicates whether it is the momentum or position space delta function. In general we will also neglect the hat in the Fourier transform. To avoid confusion, we follow the convention that

the variables x, y, z etc denote position space, the variables p, q, k, u, v etc. denote momentum space. We also simplify the notation

$$\sum_p := \sum_{p \in \Lambda^*}$$

i.e. momentum summation is always on the Λ^* . it will be more convenient to redefine the bosonic operators as

$$a_k \rightarrow \frac{1}{\sqrt{|\Lambda|}} a_k, \quad a_k^\dagger \rightarrow \frac{1}{\sqrt{|\Lambda|}} a_k^\dagger,$$

(without changing the notation) i.e. from now on we assume that

$$[a_p, a_q^\dagger] = a_p a_q^\dagger - a_q^\dagger a_p = \begin{cases} 1 & \text{if } p = q \\ 0 & \text{otherwise.} \end{cases}$$

Thus our Hamiltonian in the Fork space $\mathcal{F}_\Lambda = \oplus_N \mathcal{H}_{N,\Lambda}$, is given by

$$H_\Lambda = \sum_p p^2 a_p^\dagger a_p + \frac{1}{|\Lambda|} \sum_{p,q,u} V_u a_p^\dagger a_q^\dagger a_{p-u} a_{q+u}, \quad (2.2)$$

2.2 Free energy

The free energy per unit volume of the system at temperature $T = \beta^{-1} > 0$, density $\varrho = N/|\Lambda| > 0$ in cubic box Λ is given by

$$f(\varrho, \Lambda, \beta) = -\frac{1}{|\Lambda|\beta} \ln (\text{Tr}_{\mathcal{H}_{N,\Lambda}} \text{Exp}(-\beta H_{N,\Lambda})), \quad (2.3)$$

Let $f^P(\varrho, \Lambda, \beta)$ and $f^D(\varrho, \Lambda, \beta)$ denote the free energy per unit volume of the system with periodic or Dirichlet boundary conditions. Furthermore, we denote by $f(\varrho, \beta)$ the free energy (per unit volume) in thermodynamic limit, i.e., $|\Lambda|, N \rightarrow \infty$ with $\varrho = N/|\Lambda|$ fixed, i.e.,

$$f^{P(D)}(\varrho, \beta) \equiv \lim_{|\Lambda| \rightarrow \infty} f^{P(D)}(\varrho, \Lambda, \beta) \quad (2.4)$$

As mentioned in the introduction, in this paper, we give a upper bound on the leading order correction of $f(\varrho, \beta)$, compared with a ideal gas, in the case that $a^3 \varrho$ is small and $\lim_{\varrho \rightarrow 0} \beta \varrho^{2/3} \in (0, \infty)$. We note that $a^3 \varrho$ and $\beta \varrho^{2/3}$ are dimensionless quantities.

2.3 Ideal Bose gas in the Thermodynamic Limit

In this section, we review some well known results. In the case of vanishing interaction potential ($V = 0$), the free energy per unit volume in the thermodynamic limit can be evaluated explicitly. Let ζ denote the Riemann zeta function. It is well known that when $\varrho^{2/3}\beta \geq (4\pi)^{-1}\zeta(3/2)^{2/3}$, i.e., ϱ is greater than critical density ϱ_c ,

$$\varrho \geq \varrho_c \equiv (4\pi\beta)^{-3/2}\zeta(3/2) \quad (2.5)$$

the free energy in the thermodynamic limit is given as

$$f_0^{D(P)}(\varrho, \beta) = \frac{1}{(2\pi)^3\beta} \int_{\mathbb{R}^3} \ln(1 - e^{-\beta p^2}) d^3p \quad (2.6)$$

On the other hand, when $\varrho \leq \varrho_c$,

$$f_0^{D(P)}(\varrho, \beta) = \varrho\mu + \frac{1}{(2\pi)^3\beta} \int_{\mathbb{R}^3} \ln(1 - e^{-\beta(p^2-\mu)}) d^3p \quad (2.7)$$

Here $\mu(\varrho, \beta) < 0$ is determined by

$$\varrho = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{1}{e^{\beta(p^2-\mu)} - 1} d^3p \quad (2.8)$$

Note: when $\varrho \geq \varrho_c$, $\mu(\varrho, \beta)$ is defined as zero.

It is easy to see the scaling relation: $f_0^{D(P)}(\varrho, \beta) = \varrho^{5/3} f_0^{D(P)}(\varrho^{2/3}\beta, 1)$ and the ratio ϱ_c/ϱ only depends on dimensionless quantity $\varrho^{2/3}\beta$, i.e.,

$$\varrho_c/\varrho = (4\pi)^{-3/2}\zeta(3/2)(\varrho^{2/3}\beta)^{-3/2} \quad (2.9)$$

Let $\beta(\varrho)$ be a function of ϱ , we define $R[\beta]$ as the ratio ϱ_c/ϱ in the limit $\varrho \rightarrow 0$,

$$R[\beta] \equiv \lim_{\varrho \rightarrow 0} \varrho_c(\beta)/\varrho = \lim_{\varrho \rightarrow 0} (4\pi)^{-3/2}\zeta(3/2) \left(\varrho^{2/3}\beta(\varrho) \right)^{-3/2} \quad (2.10)$$

2.4 Scattering length

In this paper, we use the standard definition of scattering length, as in [14], [7], [4], [17], [5], [16], [15]. Let $1 - w$ be the zero energy scattering solution, i.e.,

$$-\Delta(1 - w) + V(1 - w) = 0 \quad (2.11)$$

with $0 \leq w < 1$ and $w(x) \rightarrow 0$ as $|x| \rightarrow \infty$. Then the scattering length is given by the formula

$$a := \frac{1}{4\pi} \int_{\mathbb{R}^3} V(x)(1 - w(x))dx \quad (2.12)$$

With (2.11), we have, for $p \neq 0$,

$$w_p = [V(1 - w)]_p |p|^{-2}, \quad (2.13)$$

This implies that if V is smooth, then

$$\left| \frac{dw_p}{dp} \right| \leq \text{const.} (|p|^{-3} + |p|^{-2}) \quad (2.14)$$

On the other hand, because $V(1 - w) \geq 0$, so for $\forall p$,

$$|[V(1 - w)]_p| \leq \int V(1 - w).$$

Then with (2.12), i.e., $\int V(1 - w)$ is equal to $4\pi a$, we obtain the following bound on w_p

$$|w_p| \leq 4\pi a |p|^{-2} \quad (2.15)$$

2.5 Main results

THEOREM 1. *In the temperature region where $\lim_{\varrho \rightarrow 0} \varrho^{2/3} \beta(\varrho) \in (0, \infty)$ and in thermodynamic limit, for fixed scattering length a , we have the following upper bound on the free energy difference per volume between interacting Bose gas $f^D(\varrho, \beta)$ and ideal Bose gas $f_0^D(\varrho, \beta)$,*

$$\overline{\lim}_{\varrho \rightarrow 0} (f^D(\varrho, \beta) - f_0^D(\varrho, \beta)) \varrho^{-2} \leq 4\pi a (2 - [1 - R[\beta]]_+^2), \quad (2.16)$$

where $R[\beta]$ is defined in (2.10) as the ratio ϱ_c/ϱ in the limit $\varrho \rightarrow 0$.

It is well known that the effect of boundary condition in the thermodynamic limit is negligible, i.e.,

$$f_0(\varrho, \beta) \equiv f_0^D(\varrho, \beta) = f_0^N(\varrho, \beta) \text{ and } f(\varrho, \beta) \equiv f^D(\varrho, \beta) = f^N(\varrho, \beta) \quad (2.17)$$

So with the lower bound on $f^N(\varrho, \beta)$ in Seiringer's work in [15], we can obtain the following result.

COROLLARY 1. *With the assumption of Theorem 1, we have:*

$$\lim_{\varrho \rightarrow 0} (f^D(\varrho, \beta) - f_0^D(\varrho, \beta)) \varrho^{-2} = 4\pi a (2 - [1 - R[\beta]]_+^2), \quad (2.18)$$

3 Basic strategy

3.1 Reduction to Small Torus with Periodic Boundary Conditions

To obtain the upper bound to the free energy, we can use the variational principle, which states that, for any state $\Gamma^{D(P)}$ ($\mathcal{H}_N \rightarrow \mathcal{H}_N$) satisfying Dirichlet (Periodic) boundary condition, the following inequality holds.

$$f^{D(P)}(\varrho, \Lambda, \beta) \leq \frac{1}{|\Lambda|} \text{Tr}_{\mathcal{H}_{N,\Lambda}} H_{N,\Lambda} \Gamma^{D(P)} - \frac{1}{|\Lambda|\beta} S(\Gamma^{D(P)}) \quad (3.1)$$

Here, $S(\Gamma) = -\text{Tr} \Gamma \ln \Gamma$ denotes the von-Neumann entropy. Hence, to prove Theorem 1, one only needs to construct a trial states $\Gamma^D(\varrho, \Lambda, \beta)$ satisfying Dirichlet boundary condition and the following inequality:

$$\begin{aligned} & \overline{\lim}_{\varrho \rightarrow 0} \overline{\lim}_{|\Lambda| \rightarrow \infty} \left(\frac{1}{|\Lambda|} \text{Tr} H_{N,\Lambda} \Gamma^D - \frac{1}{|\Lambda|\beta} S(\Gamma^D) - f_0^D(\varrho, \beta) \right) \varrho^{-2} \\ & \leq 4\pi a(2 - [1 - R[\beta]]_+^2) \end{aligned} \quad (3.2)$$

Furthermore, the proper trial states in thermodynamic limit ($\Lambda \rightarrow \infty$) can be constructed by duplicating the proper trial states in the *small* box ($\Lambda = \varrho^{-c}, c > 0$) with Dirichlet boundary condition. Hence, the following Proposition one implies our main result Theorem one.

Proposition 1. *In the temperature region where $\lim_{\varrho \rightarrow 0} \varrho^{2/3} \beta(\varrho) \in (0, \infty)$, for fixed scattering length a , there exist Λ , $|\Lambda| \geq \varrho^{-41/20}$ and trial states $\Gamma^D(\varrho, \Lambda, \beta)$ satisfying Dirichlet boundary condition and the following inequality, (Here $N = |\Lambda|\varrho$)*

$$\begin{aligned} & \overline{\lim}_{\varrho \rightarrow 0} \left(\frac{1}{|\Lambda|} \text{Tr} H_{N,\Lambda} \Gamma^D - \frac{1}{|\Lambda|\beta} S(\Gamma^D) - f_0^D(\varrho, \beta) \right) \varrho^{-2} \\ & \leq 4\pi a(2 - [1 - R(\beta)]_+^2), \end{aligned} \quad (3.3)$$

where $R(\beta)$ is defined in (2.10).

On the other hand, the next lemma shows that a Dirichlet boundary condition trial state with correct free energy can be obtained from a periodic one.

Lemma 1. *Let the volume $|\Lambda|$ be equal to $\varrho^{-41/20}$. In temperature region of theorem one, if*

$$f^P(\varrho, \Lambda, \beta) \leq \text{const. } \varrho^{5/3}, \quad (3.4)$$

then for revised box Λ^* and density ϱ^* :

$$|\Lambda^*| \equiv |\Lambda|(1 + 2\varrho^{41/120})^3, \quad \varrho^* \equiv \varrho(1 + 2\varrho^{41/120})^{-3}, \quad (3.5)$$

we have $f^D(\varrho^*, \Lambda^*, \beta)$ bounded from above as follows

$$\overline{\lim}_{\varrho \rightarrow 0} (f^D(\varrho^*, \Lambda^*, \beta) - f^P(\varrho, \Lambda, \beta)) \varrho^{-2} \leq 0 \quad (3.6)$$

We note: $|\Lambda^*| \geq (\varrho^*)^{-41/120}$. The construction of a periodic trial state yielding the correct free energy upper bound is the core of this paper. We state it as the following theorem, which gives the upper bound on $f^P(\varrho, \Lambda, \beta)$ in (3.4) and (3.6).

THEOREM 2. Assume $\lim_{\varrho \rightarrow 0} \varrho^{2/3}\beta \in (0, \infty)$. For $|\Lambda| = \varrho^{-41/20}$, $N = |\Lambda|\varrho$, there exists trial state $\Gamma(\varrho, \Lambda, \beta)$ satisfying

$$\overline{\lim}_{\varrho \rightarrow 0} \left(\frac{1}{|\Lambda|} \text{Tr } H_N \Gamma - \frac{1}{|\Lambda|\beta} S(\Gamma) - f_0^P(\varrho, \beta) \right) \varrho^{-2} \leq 4\pi a(2 - [1 - R[\beta]]_+^2) \quad (3.7)$$

It implies

$$\overline{\lim}_{\varrho \rightarrow 0} (f^P(\varrho, \Lambda, \beta) - f_0^P(\varrho, \beta)) \varrho^{-2} \leq 4\pi a(2 - [1 - R[\beta]]_+^2) \quad (3.8)$$

3.2 Proof of Proposition 1

With Lemma 1 and Theorem 2, we can prove Proposition 1 as follows.

Proof. Using the temperature function β in the assumption of Proposition 1, we define a new temperature function $\tilde{\beta}$ as follows

$$\tilde{\beta}(\varrho) = \beta(\varrho^*), \quad (3.9)$$

where $\varrho^* = \varrho(1 + 2\varrho^{41/120})^{-3}$, as in (3.5).

Insert the result in Theorem 2 into Lemma 1. With the definition of Λ^* , ϱ^* in Lemma 1(3.5), we obtain in the inverse temperature $\tilde{\beta}(\varrho)$,

$$\overline{\lim}_{\varrho \rightarrow 0} (f^D(\varrho^*, \Lambda^*, \tilde{\beta}) - f_0^P(\varrho, \tilde{\beta})) \varrho^{-2} \leq 4\pi a(2 - [1 - R[\tilde{\beta}]]_+^2). \quad (3.10)$$

Using the well known results of ideal Bose gases:

$$f_0^P(\varrho, \tilde{\beta}) = f_0^D(\varrho, \tilde{\beta}) = f_0^D(\varrho^*, \tilde{\beta})(1 + o(\varrho^{1/3})), \quad (3.11)$$

we can replace $f_0^P(\varrho, \tilde{\beta})$ in (3.10) with $f_0^D(\varrho^*, \tilde{\beta})$, i.e.,

$$\overline{\lim}_{\varrho \rightarrow 0} (f^D(\varrho^*, \Lambda^*, \tilde{\beta}) - f_0^D(\varrho^*, \tilde{\beta})) \varrho^{-2} \leq 4\pi a(2 - [1 - R[\tilde{\beta}]]_+^2). \quad (3.12)$$

Then by (3.9), we obtain $R[\beta] = R[\tilde{\beta}]$, so

$$\begin{aligned} \overline{\lim}_{\varrho \rightarrow 0} (f^D(\varrho^*, \Lambda^*, \beta(\varrho^*)) - f_0^D(\varrho^*, \beta(\varrho^*))) \varrho^{-2} &\leq 4\pi a(2 - [1 - R[\beta]]_+^2) \\ &= 4\pi a(2 - [1 - R[\tilde{\beta}]]_+^2) \end{aligned}$$

At last, with the facts: $\Lambda^* \geq (\varrho^*)^{-\frac{41}{20}}$ and the limit $\varrho \rightarrow 0$ is equivalent to the limit $\varrho^* \rightarrow 0$, we arrive at the desired result (3.3). \blacksquare

3.3 Reduction to Pure States

The Lemma 1 can be proved with standard method as in [16] and we leave the proof in section 12.1. In this subsection, we introduce the basic strategy of proving Theorem 2. With the assumption of Theorem 2, we have

$$\Lambda = [0, L]^3, L = \varrho^{-\frac{41}{60}}, N = \varrho^{-\frac{21}{20}} \text{ and } \lim_{\varrho \rightarrow 0} \varrho^{2/3} \beta \in (0, \infty). \quad (3.13)$$

We first identify four regions in the momentum space Λ^* which are relevant to the construction of the trial state: P_0 for the condensate, P_L for the low momenta, which are of the order $\varrho^{1/3}$; P_H for momenta of order one, and P_I the region between P_0 and P_L .

DEFINITION 1. *Definitions of P_0 , P_I , P_L and P_H*

Define four subsets of momentum space $\Lambda^ = (2\pi L^{-1}\mathbb{Z})^3$: P_0 , P_I , P_L and P_H as follows.*

$$\begin{aligned} P_0 &\equiv \{p = 0\} \\ P_I &\equiv \{p \in \Lambda^* | 0 < |p| < \varepsilon_L \varrho^{1/3}\} \\ P_L &\equiv \{p \in \Lambda^* | \varepsilon_L \varrho^{1/3} \leq |p| \leq \eta_L^{-1} \varrho^{1/3}\} \\ P_H &\equiv \{p \in \Lambda^* | \varepsilon_H \leq |p| \leq \eta_H^{-1}\}, \end{aligned} \quad (3.14)$$

where the parameters are chosen as follows

$$\varepsilon_L, \eta_L, \varepsilon_H, \eta_H \equiv \varrho^\eta \text{ and } \eta \equiv 1/200 \quad (3.15)$$

We remark that the momenta between P_L and P_H are irrelevant to our construction.

DEFINITION 2. *Definition of \widetilde{M} , M and N_α*

Let P denote $P_0 \cup P_L \cup P_I \cup P_H$. We define \widetilde{M} as the set of all functions $\alpha : P \rightarrow \mathbb{N} \cup 0$ such that

$$\sum_{k \in P} \alpha(k) = N \quad (3.16)$$

For any $\alpha \in \widetilde{M}$, denote by $|\alpha\rangle \in \mathcal{H}_{N,\Lambda}$ the unique state (in this case, an N -particle wave function) defined by the map α

$$|\alpha\rangle = C \prod_{k \in P} (a_k^\dagger)^{\alpha(k)} |0\rangle,$$

where the positive constant C is chosen so that $|\alpha\rangle$ is L_2 normalized.

Moreover, we define M as the following subset of \widetilde{M}

$$M \equiv \{\alpha \in \widetilde{M} | \text{supp}(\alpha) \subset P_0 \cup P_I \cup P_L \text{ and } \alpha(k) \leq m_c \text{ for } \forall k \in P_L\}, \quad (3.17)$$

where m_c is defined as

$$m_c \equiv \varrho^{-3\eta} = \varrho^{-3/200} \quad (3.18)$$

Clearly, we have

$$a_k^\dagger a_k |\alpha\rangle = \alpha(k) |\alpha\rangle, \quad \forall k \in P \quad (3.19)$$

With the definition of P_0 , P_L , P_I and P_H , for $\alpha \in M$, we define $N_\alpha \in \mathbb{R}$ as

$$N_\alpha \equiv \alpha(0)\alpha(0) + \sum_{u,v \in P_L \cup P_0, u \neq \pm v} 2\alpha(u)\alpha(v) \quad (3.20)$$

To prove Theorem 2, first, we show that, with α 's in M (3.17), we can construct a trial state Γ_0 satisfying (3.7), but with wrong coefficient in RHS.

Lemma 2. For $\Lambda = [0, L]^3$, $L = \varrho^{-\frac{41}{60}}$, $N = \varrho^{-\frac{21}{20}}$ and $\lim_{\varrho \rightarrow 0} \varrho^{2/3} \beta \in (0, \infty)$. There exists a state $\Gamma_0(\varrho, \beta)$ having the form:

$$\Gamma_0 = \sum_{\alpha \in M} g_\alpha(\varrho, \beta) |\alpha\rangle \langle \alpha|, \quad g_\alpha(\varrho, \beta) \in \mathbb{R} \quad (3.21)$$

and satisfying (It is $V_0 = \int V(x) dx^3$, not $4\pi a$ in the rhs.)

$$\overline{\lim}_{\varrho \rightarrow 0} \left(\frac{1}{|\Lambda|} \text{Tr } H_N \Gamma_0 - \frac{1}{|\Lambda| \beta} S(\Gamma_0) - f_0(\varrho, \beta) \right) \varrho^{-2} \leq V_0 (2 - [1 - R[\beta]]_+^2) \quad (3.22)$$

Furthermore, with N_α defined in (3.20), the coefficient function g_α satisfies

$$\lim_{\varrho \rightarrow 0} \sum_{\alpha \in M} N^{-2} N_\alpha g_\alpha = 2 - [1 - R(\beta)]_+^2 \quad (3.23)$$

We remark: actually Γ_0 is very close to Γ_I the canonical Gibbs state of ideal Bose gases. The state $\Gamma_0(\varrho, \beta)$ satisfies (3.22), but for most potential $V(x)$, $V_0 = \int V(x) dx^3$ is strictly larger than $4\pi a$. So we need to improve Γ_0 . To do that, we need to replace $|\alpha\rangle$'s in M with some Ψ_α 's. To introduce Ψ_α , we start with dividing the momentum space as follows.

DEFINITION 3. *Definitions of $B_H(u)$, $B_L(u)$*

Let $\varkappa_L, \varkappa_H > 0$. Divide P_L and P_H (3.14) into small boxes (could be non-rectangular box) s.t. the sides of the boxes are about ϱ^{\varkappa_L} and ϱ^{\varkappa_H} . We denote the box containing u as $B_H(u)$ when $u \in P_H$ or $B_L(u)$ when $u \in P_L$.

DEFINITION 4. *Definition of \widetilde{M}_α*

For any $\alpha \in M$, we define \widetilde{M}_α as the set of the β 's in \widetilde{M} (Def. 2) that

1. If $k \in P_I$, then $\beta(k) = \alpha(k)$.
2. There is **at most** one k in each B_L or B_H satisfying $\beta(k) \neq \alpha(k)$.
3. If $\beta(k) \neq \alpha(k)$, then

$$\begin{aligned} \beta(k) &= \alpha(k) - 1, & \text{for } k \in P_L \\ \beta(k) &= \alpha(k) + 1 = 1, & \text{for } k \in P_H \end{aligned} \quad (3.24)$$

For each $\alpha \in M$, we will construct a normalized pure state Ψ_α , which is linear combination of $\beta \in \widetilde{M}_\alpha$, i.e.,

$$|\Psi_\alpha\rangle = \sum_{\beta \in \widetilde{M}_\alpha} f_\alpha(\beta) |\beta\rangle, \quad \sum_{\beta \in \widetilde{M}_\alpha} |f_\alpha(\beta)|^2 = 1 \quad (3.25)$$

To prove Theorem 2, i.e., to improve the Γ_0 in Lemma 2, we choose the correct trial state Γ as follows:

$$\Gamma = \sum_{\alpha \in M} g_\alpha |\Psi_\alpha\rangle \langle \Psi_\alpha|, \quad (3.26)$$

where we choose g_α in (3.21) and Ψ_α in (3.25).

With proper \varkappa_L and \varkappa_H , ΔS the entropy difference between Γ_0 in (3.21) and Γ in (3.26) can be proved to be much less than $\Lambda \varrho^2$.

Lemma 3. *Let $\Lambda = \varrho^{-41/20}$, $\varkappa_L \leq 5/9$ and $\varkappa_H \leq 2/9$. Then for any $\{\Psi_\alpha, \alpha \in M\}$ having the form (3.25), we have*

$$\overline{\lim}_{\varrho \rightarrow 0} [-S(\Gamma) - (-S(\Gamma_0))] (\Lambda \varrho^2)^{-1} = 0 \quad (3.27)$$

We remark: the assumptions $\varkappa_L \leq 5/9$ and $\varkappa_H \leq 2/9$ implies:

$$\varrho^{1-4\eta-3\varkappa_L} + \varrho^{-4\eta-3\varkappa_H} \ll N \varrho^{1/3}. \quad (3.28)$$

In the next theorem, we will show, for each $\alpha \in M$, there exists a pure state Ψ_α . Comparing with $|\alpha\rangle$, the new pure state $|\Psi_\alpha\rangle$ lowers the total energy by about $(V_0 - 4\pi a)N_\alpha \Lambda^{-1}$. The construction of the pure state yielding the correct total energy is the core of the proof for theorem 2.

THEOREM 3. *Let $1/2 \geq \kappa_L \geq 4/9$ and $\kappa_H \geq 1/9$. For any $\alpha \in M$, there exists Ψ_α having the form (3.25) and satisfying:*

$$\langle \Psi_\alpha | H_N | \Psi_\alpha \rangle - \langle \alpha | H_N | \alpha \rangle + (V_0 - 4\pi a) N_\alpha \Lambda^{-1} \leq \varepsilon_\rho \varrho^2 \Lambda$$

where the ε_ρ is independent of α and $\lim_{\rho \rightarrow 0} \varepsilon_\rho = 0$.

3.4 Proof of Theorem 2

Proof. Let $1/2 \geq \kappa_L \geq 4/9$ and $2/9 \geq \kappa_H \geq 1/9$. We choose trial state Γ (3.26) with g_α in Lemma 2 (3.21) and Ψ_α 's in Theorem 3. Then Combine Theorem 3, Lemma 3 and Lemma 2. \blacksquare

This paper is organized as follows: In Section 4, we rigorously define Ψ_α 's and the trial state Γ . In Section 5, we outline the Lemmas needed to prove Theorem 3. In Section 6, we estimate the number of particles in the condensate and various momentum regimes. These estimates are the building blocks for all other estimates later on. The kinetic energy is estimated in Section 7 and the potential energy is estimated in Section 8-11. Finally in Section 12, we prove Lemma 1, 2, 3.

4 Definition of the trial pure states Ψ_α 's

In this section, we give a formal definition of the trial pure state Ψ_α 's for Theorem 3. We start with defining an (pair creation) operator $A_{p,q}^{u,v}$:

$$A_{p,q}^{u,v} : \widetilde{M} \rightarrow \widetilde{M}, \quad u, v \in P_0 \cup P_L, \quad p, q \in P_H \quad \text{and} \quad u + v = p + q \quad (4.1)$$

For $\beta \in \widetilde{M}$,

$$|A_{p,q}^{u,v} \beta\rangle = C a_p^\dagger a_q^\dagger a_u a_v |\beta\rangle, \quad (4.2)$$

where C is positive normalization constant. The operator $A_{p,q}^{u,v}$ annihilates two particles with momenta in P_L or P_0 and creates two particles with momenta in P_H . We note: the total momentum is conserved.

For simplicity, the pure trial state Ψ_α will be of the form $\sum_{\beta \in M_\alpha} f_\alpha(\beta) |\beta\rangle$ where f_α is supported in $M_\alpha \subset \widetilde{M}_\alpha$ which we now define.

DEFINITION 5. *Definition of nontrivial subset in P_L and M_α*

Let A be subset of P_L , it is called non-trivial when

1. *If $u_1, u_2 \in A$, $u_i \neq u_j$ ($i \neq j$), then $u_1 + u_2 \neq 0$*

2. If $u_1, u_2, u_3 \in A$, $u_i \neq u_j (i \neq j)$, then $u_1 + u_2 \neq u_3$
3. If $u_1, u_2, u_3, u_4 \in A$, $u_i \neq u_j (i \neq j)$, then $u_1 + u_2 \neq u_3 + u_4$.

Then recall \widetilde{M}_α in Def. 3. For $\alpha \in M$, we define the subset $M_\alpha \subset \widetilde{M}_\alpha$ as the smallest set with the following properties.

1. Denote the following subset of P_L as $P_L(\gamma, \alpha)$,

$$P_L(\gamma, \alpha) \equiv \{u \in P_L : \gamma(u) < \alpha(u)\}. \quad (4.3)$$

$P_L(\gamma, \alpha)$ is non-trivial, for any $\gamma \in M_\alpha$

2. $\alpha \in M_\alpha$
3. If $\beta \in M_\alpha$ and $\gamma = A_{p,-p}^{0,0}\beta \in \widetilde{M}_\alpha$, then $\gamma \in M_\alpha$.
4. If $\beta \in M_\alpha$, $\gamma = A_{p,q}^{u,v}\beta \in \widetilde{M}_\alpha$ and

- (a) $P_L(\gamma, \alpha)$ is non-trivial
- (b) $\beta(-p) = \alpha(-p)$, $\beta(-q) = \alpha(-q)$

then $\gamma \in M_\alpha$.

Note: The set M_α is unique since the intersection of two such sets M_{α_1} and M_{α_2} satisfies all four conditions. The properties of non-trivial set have no physical meaning, but they can simplify our proof and calculation.

We collect a few obvious properties of the elements in M_α into the next lemma.

Lemma 4. By the definition of M_α , any $\beta \in M_\alpha$ has the following form:

$$\beta = \prod_{i=1}^m \mathcal{A}_{k_{2i-1}, k_{2i}}^{u_{2i-1}, u_{2i}} \prod_{j=1}^n \mathcal{A}_{p_j, -p_j}^{0,0} \alpha, \quad (4.4)$$

where $u_i \in P_L \cup P_0$, $k_i \in P_H$ for $i = 1, \dots, 2m$ and $p_j \in P_H$ for $j = 1, \dots, n$. And

$$p_i \neq \pm p_j, \quad k_i \neq \pm k_j \quad \text{for } i \neq j \quad \text{and} \quad k_i \neq \pm p_j \quad \text{for } \forall i, j \quad (4.5)$$

On the other hand, if $\{u_i, (i = 1, \dots, 2m)\} \cap P_L$ is a non-trivial set of P_L , any β with form (4.4) and (4.5) belongs to M_α . Furthermore, one can change the order of the \mathcal{A} 's in (4.4). With the fact that the subset of non-trivial

subset of P_L is still non-trivial, we can see, if β belongs to M_α and has the form (4.4) and (4.5), then

$$\prod_{i \in A} \mathcal{A}_{k_{2i-1}, k_{2i}}^{u_{2i-1}, u_{2i}} \prod_{j \in B} \mathcal{A}_{p_j, -p_j}^{0,0} \alpha \in M_\alpha \quad (4.6)$$

Here A, B are any subsets of $\{1, \dots, m\}$ and $\{1, \dots, n\}$

DEFINITION 6. The Pure Trial State Ψ_α

Recall function $(1-w)$ is the zero energy scattering solution of potential V , as in (2.11). Define the pure trial state Ψ_α as

$$|\Psi_\alpha\rangle \equiv \sum_{\beta \in M_\alpha} f_\alpha(\beta) |\beta\rangle \quad (4.7)$$

where the coefficient $f_\alpha(\beta)$'s are given by

$$f_\alpha(\beta) = C_\alpha \sqrt{\frac{|\Lambda|^{\beta(0)}}{\beta(0)!}} \prod_{k \in P_H}^{\beta(k) > 0} \sqrt{-w_k} \prod_{k \in P_H}^{\beta(k) > \beta(-k)} \sqrt{2} \prod_{u \in P_L}^{\beta(u) < \alpha(u)} \sqrt{\frac{\alpha(u)}{|\Lambda|}} \quad (4.8)$$

Here we follow the convention $\sqrt{x} = \sqrt{|x|}i$ for $x < 0$. For convenience, we define $f(\beta) = 0$ for $\beta \notin M_\alpha$. The constant C_α is chosen so that Ψ_α is L_2 normalized, i.e.,

$$\langle \Psi_\alpha | \Psi_\alpha \rangle = 1, \text{ i.e., } \sum_{\beta \in M_\alpha} |f_\alpha(\beta)|^2 = 1$$

With choosing f_α as above, we can obtain a few obvious identities of f_α as follows.

Lemma 5. 1. If $k \in P_H$ and $\beta \in M_\alpha$, $\mathcal{A}_{k, -k}^{0,0} \beta \in M_\alpha$, then

$$f(\mathcal{A}_{k, -k}^{0,0} \beta) = (-w_k) \sqrt{\frac{\beta(0)}{|\Lambda|}} \sqrt{\frac{\beta(0)-1}{|\Lambda|}} f_\alpha(\beta) \quad (4.9)$$

2. If $u_1, u_2 \in P_L$, $u_2 = \pm u_1$ or $u_2 \in B_L(u_1)$, $k_1, k_2 \in P_H$ and $\beta \in M_\alpha$, then $\gamma = \mathcal{A}_{k_1, k_2}^{u_1, u_2} \beta \notin M_\alpha$, i.e., $f_\alpha(\gamma) = 0$.

3. If $u_1, u_2 \in P_L \cup P_0$ and $u_2 \neq \pm u_1$, $k_1, k_2 \in P_H$, $\beta \in M_\alpha$ and $\mathcal{A}_{k_1, k_2}^{u_1, u_2} \beta \in M_\alpha$, then when $\beta(-p) = \alpha(-p)$ and $\beta(-q) = \alpha(-q)$, we have

$$f(\mathcal{A}_{k_1, k_2}^{u_1, u_2} \beta) = 2\sqrt{-w_{k_1}} \sqrt{-w_{k_2}} \sqrt{\frac{\beta(u_1)}{|\Lambda|}} \sqrt{\frac{\beta(u_2)}{|\Lambda|}} f(\beta) \quad (4.10)$$

when $\beta(-p) \neq \alpha(-p)$ or $\beta(-q) \neq \alpha(-q)$, we have

$$\left| f(\mathcal{A}_{k_1, k_2}^{u_1, u_2} \beta) \right| \leq \left| \sqrt{w_{k_1}} \sqrt{w_{k_2}} \sqrt{\frac{\beta(u_1)}{|\Lambda|}} \sqrt{\frac{\beta(u_2)}{|\Lambda|}} f(\beta) \right| \quad (4.11)$$

Again the result 2 in Lemma 5 has no physical meaning, but it can simplify our proof.

At last we give a brief explanation why we mentioned in introduction that Γ the trial state is very close to (1.5). With the definition of $\mathcal{A}_{k_1, k_2}^{u_1, u_2}$, we can see that Ψ_α almost equals to

$$\Psi_\alpha \sim P_{(\varrho^{1/3}, \varrho^{1/3})} P_{(0, \varrho^{1/3})} P_{(0,0)} |\alpha\rangle \quad (4.12)$$

where

$$\begin{aligned} P_{(0,0)} &= \exp \left[\sum_k -w_k a_k^\dagger a_{-k}^\dagger a_0 a_0 \right] \\ P_{(0, \varrho^{1/3})} &= \exp \left[\sum_k \sum_{v \sim \varrho^{1/3}} -2 \sqrt{w_{k+v/2} w_{-k+v/2}} a_{k+v/2}^\dagger a_{-k+v/2}^\dagger a_v a_0 \right] \\ P_{(\varrho^{1/3}, \varrho^{1/3})} &= \exp \left[\sum_k \sum_{u \neq v \sim \varrho^{1/3}} -\sqrt{w_{k+\frac{v+u}{2}} w_{-k+\frac{v+u}{2}}} a_{k+\frac{v+u}{2}}^\dagger a_{-k+\frac{v+u}{2}}^\dagger a_v a_u \right] \end{aligned} \quad (4.13)$$

On the other hand the trial state Γ has the form $\sum_\alpha g_\alpha |\Psi_\alpha\rangle \langle \Psi_\alpha|$ and $\Gamma_0 = g_\alpha |\alpha\rangle \langle \alpha|$, so the trial state Γ is very close to the following one:

$$\Gamma \sim \left(P_{(\varrho^{1/3}, \varrho^{1/3})} P_{(0, \varrho^{1/3})} P_{(0,0)} \right) \Gamma_0 \left(P_{(\varrho^{1/3}, \varrho^{1/3})} P_{(0, \varrho^{1/3})} P_{(0,0)} \right)^\dagger \quad (4.14)$$

As mentioned before, actually Γ_0 is very close to Γ_I the Gibbs state of ideal Bose gases, so we obtain

$$\Gamma \sim \left(P_{\varrho^{1/3}, \varrho^{1/3}} P_{0, \varrho^{1/3}} P_{0,0} \right) \Gamma_I \left(P_{\varrho^{1/3}, \varrho^{1/3}} P_{0, \varrho^{1/3}} P_{0,0} \right)^\dagger \quad (4.15)$$

5 Proof of Theorem 3

Proof. Our goal is to prove

$$\langle \Psi_\alpha | H_N | \Psi_\alpha \rangle - \langle \alpha | H_N | \alpha \rangle + (V_0 - 4\pi a) N_\alpha \leq \varepsilon_\varrho \varrho^2 \Lambda \quad (5.1)$$

First we decompose the Hamiltonian H_N . By the rule 1 of the definition of \widetilde{M}_α , $\beta(k)$ is equal to $\alpha(k)$ for any $k \in P_I$ and $\beta \in M_\alpha \subset \widetilde{M}_\alpha$. So if $k_1 \in P_I$, $\beta, \gamma \in M_\alpha$ and $\langle \beta | a_{k_1}^\dagger a_{k_2}^\dagger a_{k_3} a_{k_4} | \gamma \rangle \neq 0$, then one of k_3 and k_4 must be equal to k_1 . On the other hand, since the particles with momenta in P_H are created by pair, the total number of the particles with momenta in P_H is always even. With these two results and momentum conservation, we can decompose the expectation value $\langle \Psi_\alpha | H_N | \Psi_\alpha \rangle$ as follows:

$$\langle H_N \rangle_{\Psi_\alpha} = \left\langle \sum_{i=1}^N -\Delta_i \right\rangle_{\Psi_\alpha} + \langle H_{abab} \rangle_{\Psi_\alpha} + \langle H_{\widetilde{L}\widetilde{L}} \rangle_{\Psi_\alpha} + \langle H_{\widetilde{L}H} \rangle_{\Psi_\alpha} + \langle H_{HH} \rangle_{\Psi_\alpha}, \quad (5.2)$$

where

1. H_{abab} is the part of interaction that annihilates two particles and creates the same two particles, i.e.,

$$H_{abab} = |\Lambda|^{-1} \sum_u V_0 a_u^\dagger a_u^\dagger a_u a_u + |\Lambda|^{-1} \sum_{u \neq v} (V_{u-v} + V_0) a_u^\dagger a_v^\dagger a_u a_v \quad (5.3)$$

2. $H_{\widetilde{L}\widetilde{L}}$ is the interaction between particles with momenta in $P_{\widetilde{L}}$:

$$P_{\widetilde{L}} \equiv P_0 \cup P_L \quad (5.4)$$

and

$$H_{\widetilde{L}\widetilde{L}} = |\Lambda|^{-1} \sum_{u_i \in P_{\widetilde{L}}} V_{u_3-u_1} a_{u_1}^\dagger a_{u_2}^\dagger a_{u_3} a_{u_4}, \quad (5.5)$$

where $u_1 \neq u_3$ or u_4 .

3. $H_{\widetilde{L}H}$ is the part of interaction that involves two particles with momenta in $P_{\widetilde{L}}$ and two particles with momenta in P_H i.e.,

$$\begin{aligned} H_{\widetilde{L}H} = & |\Lambda|^{-1} \sum_{u_1, u_2 \in P_{\widetilde{L}}, k_1, k_2 \in P_H} V_{u_1-k_1} a_{u_1}^\dagger a_{u_2}^\dagger a_{k_1} a_{k_2} + C.C \quad (5.6) \\ & + |\Lambda|^{-1} \sum_{u_1, u_2 \in P_{\widetilde{L}}, k_1, k_2 \in P_H} 2(V_{u_1-u_2} + V_{u_1-k_2}) a_{u_1}^\dagger a_{k_1}^\dagger a_{u_2} a_{k_2}, \end{aligned}$$

where $u_1 \neq u_2$.

4. H_{HH} is the part of interaction between particles with momenta in P_H ,

$$H_{HH} = |\Lambda|^{-1} \sum_{k_i \in P_{\widetilde{H}}} V_{k_3-k_1} a_{k_1}^\dagger a_{k_2}^\dagger a_{k_3} a_{k_4}, \quad (5.7)$$

where $k_1 \neq k_3$ or k_4 .

With these definitions, we can rewrite the total energy of $|\alpha\rangle$ as:

$$\langle \alpha | H_N | \alpha \rangle = \langle \alpha | \sum_{i=1}^N -\Delta_i | \alpha \rangle + \langle \alpha | H_{abab} | \alpha \rangle \quad (5.8)$$

The estimates for the energies of these components of in (5.2) about Ψ_α are stated as the following lemmas, which will be proved in later sections.

Lemma 6. *The total kinetic energy is bounded from above by*

$$\left\langle \sum_{i=1}^N -\Delta_i \right\rangle_{\Psi_\alpha} - \left\langle \sum_{i=1}^N -\Delta_i \right\rangle_{\alpha} - \|\nabla w\|_2^2 N_\alpha |\Lambda|^{-1} \leq \varepsilon_1 \varrho^2 \Lambda, \quad (5.9)$$

where ε_1 is independent of ϱ and $\lim_{\varrho \rightarrow 0} \varepsilon_1 = 0$.

Lemma 7. *The expectation value of H_{abab} is bounded above by,*

$$\langle H_{abab} \rangle_{\Psi_\alpha} - \langle H_{abab} \rangle_{\alpha} \leq \varrho^{11/4} \Lambda \quad (5.10)$$

Lemma 8. *The expectation value of $H_{\tilde{L}\tilde{L}}$ is bounded above by,*

$$\langle H_{\tilde{L}\tilde{L}} \rangle_{\Psi_\alpha} \leq \varrho^{11/4} \Lambda \quad (5.11)$$

Lemma 9. *The expectation value of $H_{\tilde{L}H}$ is bounded above by,*

$$\langle H_{\tilde{L}H} \rangle_{\Psi_\alpha} + N_\alpha |\Lambda|^{-1} \|2Vw\|_1 \leq \varepsilon_2 \varrho^2 \Lambda, \quad (5.12)$$

where ε_2 is independent of α and $\lim_{\varrho \rightarrow 0} \varepsilon_2 = 0$.

Lemma 10. *The expectation value of H_{HH} is bounded above by,*

$$\langle H_{HH} \rangle_{\Psi_\alpha} - N_\alpha |\Lambda|^{-1} \|Vw^2\|_1 \leq \varepsilon_3 \varrho^2 \Lambda, \quad (5.13)$$

where ε_3 is independent of α and $\lim_{\varrho \rightarrow 0} \varepsilon_3 = 0$.

On the other hand, by definition of w in (2.11) and (2.12), we have

$$\|\nabla w\|_2^2 - \|Vw\|_1 + \|Vw^2\|_1 = 0, \quad V_0 - \|Vw\|_1 = 4\pi a \quad (5.14)$$

Together with (5.8) and (5.9)-(5.13), we arrive at the desired result (5.1). ■

6 Estimates on the Numbers of Particles

The first step to prove the Lemma 6 to Lemma 10 is to estimate the particle number of Ψ_α in the condensate, P_L , P_I , and P_H . This is the main task of this section and we start with the following notations.

DEFINITION 7. Suppose $u_i \in P$ for $i = 1, \dots, s$. The expectation of the product of particle numbers with momenta u_1, \dots, u_s :

$$Q_\alpha(u_1, u_2, \dots, u_s) \equiv \left\langle \prod_{i=1}^s a_{u_i}^\dagger a_{u_i} \right\rangle_{\Psi_\alpha} = \sum_{\beta \in M_\alpha} \prod_{i=1}^s \beta(u_i) |f_\alpha(\beta)|^2 \quad (6.1)$$

DEFINITION 8. We denote by $M_\alpha(u) \subset M_\alpha$ the set of β 's satisfying $\beta(u) = \alpha(u)$, i.e.,

$$M_\alpha(u) \equiv \{\beta \in M_\alpha : \beta(u) = \alpha(u)\} \quad (6.2)$$

Furthermore, with the definition of B_L and B_H , we define $M_\alpha^B(u) \subset M_\alpha(u)$ as the intersection of $M_\alpha(v)$'s of all $v \in B_L(u)$ or $B_H(u)$, i.e.,

$$M_\alpha^B(u) \equiv \cap_{v \in B_L(u) \text{ or } B_H(u)} M_\alpha(v) \quad (6.3)$$

We can see

$$\beta \in M_\alpha^B(u) \Leftrightarrow \beta(v) = \alpha(v) \text{ for } \forall v \in B_L(u) \text{ or } B_H(u) \quad (6.4)$$

Using (3.24), we can see that, for $u \in P_L$, the expectation of the particle number $Q_\alpha(u) = \langle a_u^\dagger a_u \rangle_{\Psi_\alpha}$ is equal to $\alpha(u) - \sum_{\beta \notin M_\alpha(u)} |f(\beta)|^2$. For $k \in P_H$, the expectation value of particle number $Q_\alpha(k)$ is equal to $\sum_{\beta \notin M_\alpha(u)} |f(\beta)|^2$. The following theorem provides the main estimates on $Q_\alpha(u)$ and $Q_\alpha(k)$.

Lemma 11. For small enough ϱ , $Q_\alpha(u)$ and $Q_\alpha(k)$ can be estimated as follows ($u, u_1, u_2 \in P_L$ and $k \in P_H$)

$$Q_\alpha(k) = \sum_{\beta \notin M_\alpha(k)} |f(\beta)|^2 \leq \text{const. } \varrho^{2-4\eta}, \text{ for } k \in P_H \quad (6.5)$$

$$0 \leq \alpha(u) - Q_\alpha(u) = \sum_{\beta \notin M_\alpha(u)} |f(\beta)|^2 \leq \text{const. } \varrho^{1-4\eta}, \text{ for } u \in P_L \quad (6.6)$$

Furthermore, the probabilities of the combined cases are bounded as follows: ($u, u_1, u_2 \in P_L$ and $k \in P_H$)

$$\sum_{\beta \notin M_\alpha(u_1) \cup M_\alpha(u_2)} |f(\beta)|^2 \leq \text{const. } \varrho^{2-8\eta} \text{ when } u_1 \neq u_2 \quad (6.7)$$

$$\sum_{\beta \notin M_\alpha(u) \cup M_\alpha(k)} |f(\beta)|^2 \leq \text{const. } \varrho^{3-7\eta} |w_k| \quad (6.8)$$

Proof. Proof of Lemma 11

First, we prove (6.5) concerning $k \in P_H$. Using Lemma 4((4.4)-(4.6)), we have when $\beta(k) > 0$, there exist $\gamma \in M_\alpha$ and $u, v \in P_L \cup P_0$, $p \in P_H$ s.t.

$$\mathcal{A}_{k,p}^{u,v} \gamma = \beta \quad \text{and} \quad p = u + v - k \quad (6.9)$$

With the properties of f_α in Lemma 5((4.9)-(4.11)), $f_\alpha(\beta)$ can be bounded as

$$|f_\alpha(\beta)|^2 \leq 4\gamma(u)\gamma(v)\Lambda^{-2} |w_k w_p| |f_\alpha(\gamma)|^2. \quad (6.10)$$

Then sum up $\beta \notin M_\alpha(k)$, i.e., $\beta(k) > 0$, by summing up u, v and γ , we obtain:

$$\begin{aligned} \sum_{\beta \notin M_\alpha(k)} |f_\alpha(\beta)|^2 &\leq 4 \sum_{u,v \in P_L \cup P_0} \sum_{\gamma \in M_\alpha} \gamma(u)\gamma(v)\Lambda^{-2} |w_k w_{u+v-k}| |f_\alpha(\gamma)|^2 \\ &\leq 4\varrho^2 |w_k| \max_{p \in P_H} \{|w_p|\} \end{aligned} \quad (6.11)$$

Then, with the result in (2.15): $|w_p| \leq 4\pi a |p|^{-2}$, we obtain

$$Q_\alpha(k) \leq \text{const.} \varrho^{2-2\eta} |w_k|, \quad k \in P_H \quad (6.12)$$

Using (2.15) again, we obtain (6.5).

Then, we prove (6.6) concerning $u \in P_L$. Similarly, with Lemma 4, for any $\beta \notin M_\alpha(u)$, i.e., $\beta(u) = \alpha(u) - 1$, there exist $\gamma \in M_\alpha$ and $v \in P_L \cup P_0$, $p, k \in P_H$ s.t. (6.9) holds. This implies (6.10). Using (2.15) and $|k+p| \ll |k|$, we have

$$|w_p w_k| \leq \text{const.} |k|^{-4}, \quad \text{when } p, k \in P_H \quad \text{and} \quad |p+k| \ll |k| \quad (6.13)$$

Inserting this inequality and the bounds $\alpha(u) \leq m_c = \varrho^{-3\eta}$ into (6.10), we obtain:

$$|f_\alpha(\beta)|^2 \leq \text{const.} \varrho^{-3\eta} |k|^{-4} \gamma(v) \Lambda^{-2} |f_\alpha(\gamma)|^2 \quad (6.14)$$

Again, summing up β , with $\sum_v \gamma(v) \leq N$, we obtain (6.6) as follows

$$\sum_{\beta \notin M_\alpha(u)} |f_\alpha(\beta)|^2 \leq \sum_{v \in P_L \cup P_0} \sum_{\gamma \in M_\alpha}^{k \in P_H} \text{const.} \varrho^{-3\eta} |k|^{-4} \gamma(v) \Lambda^{-2} |f_\alpha(\gamma)|^2 \leq \varrho^{1-4\eta} \quad (6.15)$$

Then, we prove (6.7) concerning $u_1, u_2 \in P_L$. For any $\beta \notin M_\alpha(u_1) \cup M_\alpha(u_2)$, using Lemma 4, we can see that there are only two cases:

1. there exist one $\gamma \in M_\alpha$, $p_1, p_2 \in P_H$ and $\mathcal{A}_{p_1, p_2}^{u_1, u_2} \gamma = \beta$

2. there exist one $\gamma \notin M_\alpha(u_2)$, $v \in P_L \cup P_0$, $v \neq u_2$, $p_1, p_2 \in P_H$ and $\mathcal{A}_{p_1, p_2}^{u_1, v} \gamma = \beta$

As before, with the properties of f_α in Lemma 5, the bounds on $\alpha(u)$'s ($u \in P_L$) and (6.13), we have

$$\begin{aligned} \sum_{\beta \notin M_\alpha(u_1) \cup M_\alpha(u_2)} |f_\alpha(\beta)|^2 &\leq \text{const.} \sum_{\gamma \in M_\alpha} \varrho^{-7\eta} |\Lambda|^{-1} |f_\alpha(\gamma)|^2 \\ &+ \text{const.} \sum_{v \in P_L \cup P_0, \gamma \notin M_\alpha(u_2)} \varrho^{-4\eta} \gamma(v) |\Lambda|^{-1} |f_\alpha(\gamma)|^2 \end{aligned} \quad (6.16)$$

Using $\sum_v \gamma(v) \leq N$ and (6.6), we obtain (6.7).

At last, we prove (6.8) concerning $u \in P_L$ and $k \in P_k$. For any $\beta \notin M_\alpha(u) \cup M_\alpha(k)$, Using Lemma 4, we can see that there are only two cases:

1. there exist $\gamma \in M_\alpha$, $v \in P_L \cup P_0$, $p \in P_H$ and $\mathcal{A}_{p, k}^{u, v} \gamma = \beta$
2. there exist $\gamma \notin M_\alpha(u)$, $v_1, v_2 \in P_L \cup P_0$, $p \in P_H$ and $\mathcal{A}_{p, k}^{v_1, v_2} \gamma = \beta$

Summing up v , p or v_1 , v_2 , p , we obtain

$$\begin{aligned} \sum_{\beta \notin M_\alpha(k) \cup M_\alpha(u)} |f_\alpha(\beta)|^2 &\leq \text{const.} \sum_{v \in P_L \cup P_0} \sum_{\gamma} \gamma(u) \gamma(v) \Lambda^{-2} |w_k w_{u+v-k}| |f_\alpha(\gamma)|^2 \\ &+ \sum_{\gamma \notin M_\alpha(u)} 4\varrho^2 |w_k| \max_{p \in P_H} \{|w_p|\} |f_\alpha(\gamma)|^2 \end{aligned} \quad (6.17)$$

With the result in (2.15): $|w_p| \leq 4\pi a |p|^{-2}$ and $\sum_v \gamma(v) \leq N$, we have:

$$\sum_{\beta \notin M_\alpha(k) \cup M_\alpha(u)} |f_\alpha(\beta)|^2 \leq \text{const.} \gamma(u) \varrho^{1-2\eta} \Lambda^{-1} |w_k| + \sum_{\gamma \notin M_\alpha(u)} 4\varrho^{2-2\eta} |w_k| |f_\alpha(\gamma)|^2 \quad (6.18)$$

At last using (6.6) and the fact $\gamma(u) \leq \alpha(u) \leq \varrho^{-3\eta}$ and $\Lambda = \varrho^{-41/20}$, we obtain the desired result (6.8) \blacksquare

Moreover $Q_\alpha(k)$ ($k \in P_H$), has a more precise upper bound as follows.

Lemma 12. *For $k \in P_H$, and $Q_\alpha(k)$ is bounded above by:*

$$Q_\alpha(k) \leq N_\alpha \Lambda^{-2} |w_k|^2 + \varrho^{7/3-7\eta} \quad (6.19)$$

Proof. First using Lemma 4, we have that, for any $\beta \notin M_\alpha(k)$, there are two cases:

1. there exist $\gamma \in M_\alpha$, s.t., $\mathcal{A}_{-k,k}^{0,0}\gamma = \beta$
2. there exist $\gamma \in M_\alpha$, $u \neq \pm v \in P_L \cup P_0$, $p \in P_H$, s.t., $\mathcal{A}_{p,k}^{v_1,v_2}\gamma = \beta$.

So using the property of f_α in Lemma 5, $Q_\alpha(k) = \sum_{\beta \notin M_\alpha(k)} |f_\alpha(\beta)|^2$ is bounded above by

$$\sum_{\beta \notin M_\alpha(k)} |f_\alpha(\beta)|^2 \leq \alpha(0)^2 \Lambda^{-2} w_k^2 + \sum_{u,v \in P_L \cup P_0, u \neq \pm v} 2\alpha(u)\alpha(v)\Lambda^{-2} |w_k w_p| \quad (6.20)$$

where $p = u + v - k$. Since $w_p = w_{-p}$ and $|p + k| \leq 2(\varrho^{1/3-\eta})$, with (2.14), we have

$$||w_k| - |w_p|| \leq \text{const. } \varrho^{1/3-4\eta} \quad (6.21)$$

Inserting this into (6.20), we obtain

$$Q_\alpha(k) \leq N_\alpha \Lambda^{-2} w_k^2 + \varrho^{7/4-4\eta} |w_k| \quad (6.22)$$

Combine with the fact $|w_k| \leq \text{const. } \varrho^{-2\eta}$, we obtain the desired result (6.19). ■

At last, We collect a few obvious inequalities of f_α into the following lemma.

Lemma 13. : Recall the definition of $M_\alpha^B(k)$ or $M_\alpha^B(u)$ in Def. 8 (6.3), the results (6.6) and (6.5) implies:

$$\sum_{\beta \notin M_\alpha^B(k)} |f_\alpha(\beta)|^2 \leq \varrho^{2-4\eta} \Lambda \varrho^{3\kappa_H} \leq \varrho^{1/6} \text{ for } k \in P_H \quad (6.23)$$

and

$$\sum_{\beta \notin M_\alpha^B(u)} |f_\alpha(\beta)|^2 \leq \varrho^{1-4\eta} \Lambda \varrho^{3\kappa_L} = \varrho^{1/6} \text{ for } u \in P_L \quad (6.24)$$

Recall the definition of non-trivial subset of P_L in Def. 5. For any fixed $u, v \in P_L$, define $M_\alpha^{u,v} \subset M_\alpha(u) \cap M_\alpha(v)$ as follows

$$M_\alpha^{u,v} \equiv \{\beta \in M_\alpha(u) \cap M_\alpha(v) \mid \{u, v \cup P_L(\beta, \alpha)\} \text{ is non-trivial}\} \quad (6.25)$$

Here $P_L(\beta, \alpha)$ is defined in (4.3) as the set $\{u \in P_L : \beta(u) < \alpha(u)\}$. Then using (6.6) and (6.7), we have

$$\sum_{\beta \notin M_\alpha^{u,v}} |f_\alpha(\beta)|^2 \leq \varrho^{1/2} \quad (6.26)$$

Similarly, for any fixed $u \in P_L$, define $M_\alpha^{0,u} \subset M_\alpha(u)$ as follows

$$M_\alpha^{0,u} \equiv \{\beta \in M_\alpha(u) | \{u \cup P_L(\beta)\} \text{ is non-trivial}\} \quad (6.27)$$

Then using (6.6) and (6.7), we have

$$\sum_{\beta \notin M_\alpha^{0,u}} |f(\beta)|^2 \leq \varrho^{1/2} \quad (6.28)$$

At last, with (6.5) and the fact $\alpha(0) - \beta(0) \leq \sum_{k \in P_H} \beta(k)$, we have $Q_\alpha(0)$ and $Q_\alpha(0,0)$ bounded as follows

$$\alpha(0) \geq Q_\alpha(0) \geq \alpha(0) - \varrho^{5/6} N \quad (6.29)$$

and

$$[\alpha(0)]^2 \geq Q_\alpha(0,0) \geq [\alpha(0)]^2 - N^2 \varrho^{5/6} \quad (6.30)$$

7 Proof of Lemma 6

In this section, we estimate the kinetic energy of Ψ_α by proving Lemma 6.

Proof. By the definition,

$$\left\langle \sum_{i=1}^N -\Delta_i \right\rangle_{\Psi_\alpha} = \sum_{u \in P_L \cup P_I \cup P_H} u^2 Q_\alpha(u) \quad \text{and} \quad \left\langle \sum_{i=1}^N -\Delta_i \right\rangle_\alpha = \sum_{u \in P_L \cup P_I} u^2 \alpha(u) \quad (7.1)$$

On the other hand $Q_\alpha(u) \leq \alpha(u)$, for $u \in P_I \cup P_L$. So we obtain the LHS of (5.9) bounded above by

$$\begin{aligned} & \left\langle \sum_{i=1}^N -\Delta_i \right\rangle_{\Psi_\alpha} - \left\langle \sum_{i=1}^N -\Delta_i \right\rangle_\alpha - \|\nabla w\|_2^2 N_\alpha |\Lambda|^{-1} \\ & \leq \sum_{k \in P_H} k^2 Q_\alpha(k) - \|\nabla w\|_2^2 N_\alpha |\Lambda|^{-1} \end{aligned} \quad (7.2)$$

With the upper bound on $Q_\alpha(k)$ in (6.19), we have

$$(7.2) \leq N_\alpha |\Lambda|^{-1} \left| \|\nabla w\|_2^2 - \sum_{k \in P_H} |\Lambda|^{-1} k^2 |w_k|^2 \right| + \varrho^{13/6} \Lambda \quad (7.3)$$

Then using the fact $\lim_{\varrho \rightarrow 0} \left| \|\nabla w\|_2^2 - \sum_{k \in P_H} |\Lambda|^{-1} k^2 |w_k|^2 \right| = 0$, we complete the proof of Lemma 6. ■

8 Proof of Lemma 7

Proof. First we rewrite the expectation value of H_{abab} as

$$\begin{aligned}
& \langle H_{abab} \rangle_{\Psi_\alpha} \tag{8.1} \\
&= |\Lambda|^{-1} \sum_{\beta \in M_\alpha} \left(V_0 \sum_u (\beta(u)^2 - \beta(u)) + \sum_{u \neq v} (V_0 + V_{u-v}) \beta(u) \beta(v) \right) |f_\alpha(\beta)|^2 \\
&= |\Lambda|^{-1} \left(V_0(N^2 - N) + \sum_{u \neq v} V_{u-v} \beta(u) \beta(v) \right) |f_\alpha(\beta)|^2
\end{aligned}$$

On the other hand,

$$\langle H_{abab} \rangle_\alpha = |\Lambda|^{-1} \left(V_0(N^2 - N) + \sum_{u \neq v} V_{u-v} \alpha(u) \alpha(v) \right) \tag{8.2}$$

By the assumptions, V_v is positive for any $|v| \ll 1$. Together with the fact $\beta(u) \leq \alpha(u)$ for $u \in P_0 \cup P_I \cup P_L$, we have for $\beta \in M_\alpha$

$$V_{u-v} \beta(u) \beta(v) \leq V_{u-v} \alpha(u) \alpha(v), \text{ when } u, v \in P_0 \cup P_I \cup P_L \tag{8.3}$$

Using this inequality and the fact $\alpha(k) = 0$ for $k \in P_H$, we have

$$\begin{aligned}
& \langle H_{abab} \rangle_{\Psi_\alpha} - \langle H_{abab} \rangle_\alpha \\
&\leq |\Lambda|^{-1} \left(\sum_{u \notin P_H, v \in P_H} 2V_{u-v} Q_\alpha(u, v) + \sum_{u, v \in P_H} V_{u-v} Q_\alpha(u, v) \right)
\end{aligned}$$

Using the fact $|V_u|$ is no more than $|V_0|$ for any $u \in P$, we obtain:

$$\langle H_{abab} \rangle_{\Psi_\alpha} - \langle H_{abab} \rangle_\alpha \leq 2V_0 \varrho \sum_{v \in P_H} Q_\alpha(v) \leq \varrho^{11/4} \Lambda \tag{8.4}$$

For the last inequality, we used (6.5). ■

9 Proof of Lemma 8

We start the proof with the following identity for $\langle a_{u_1}^\dagger a_{u_2}^\dagger a_{u_3} a_{u_4} \rangle_{\Psi_\alpha}$.

Lemma 14. For any fixed momenta $u_{1,2,3,4}$ and $\beta \in M_\alpha$, define $T(\beta)$ to be the state

$$|T(\beta)\rangle \equiv C a_{u_1}^\dagger a_{u_2}^\dagger a_{u_3} a_{u_4} |\beta\rangle, \quad (9.1)$$

where C is the positive normalization constant when $|T(\beta)\rangle \neq 0$. Then we have

$$\langle a_{u_1}^\dagger a_{u_2}^\dagger a_{u_3} a_{u_4} \rangle_{\Psi_\alpha} = \sum_{\beta \in M_\alpha} f_\alpha(\beta) \overline{f_\alpha(T(\beta))} \sqrt{\langle \beta | a_{u_4}^\dagger a_{u_3}^\dagger a_{u_2} a_{u_1} | a_{u_1}^\dagger a_{u_2}^\dagger a_{u_3} a_{u_4} | \beta \rangle} \quad (9.2)$$

The map T depends on $u_{1,2,3,4}$ and in principle it has to carry them as subscripts. We omit these subscripts since it will be clear from the context what they are.

Proof. For any fixed $u_{1,2,3,4}$, by the definition of Ψ_α , we have

$$\langle \Psi_\alpha | a_{u_1}^\dagger a_{u_2}^\dagger a_{u_3} a_{u_4} | \Psi_\alpha \rangle = \sum_{\gamma, \beta \in M} f_\alpha(\beta) \overline{f_\alpha(\gamma)} \langle \gamma | a_{u_1}^\dagger a_{u_2}^\dagger a_{u_3} a_{u_4} | \beta \rangle \quad (9.3)$$

By definition of M_α , one can see

$$\langle \gamma | a_{u_1}^\dagger a_{u_2}^\dagger a_{u_3} a_{u_4} | \beta \rangle \neq 0 \Rightarrow \gamma = T(\beta) \quad (9.4)$$

Since $|T(\beta)\rangle$ is normalized, the identity in Lemma 14 is obvious. ■

9.1 Proof of Lemma 8

Proof. Using the fact $|V_u| \leq |V_0|$ for any $u \in \mathbb{R}^3$, we can see

$$\left| \langle H_{\tilde{L}\tilde{L}} \rangle_{\Psi_\alpha} \right| \leq |V_0 \Lambda|^{-1} \sum_{u_i \in P_L, u_1 \neq u_3, u_4} \left| \langle a_{u_1}^\dagger a_{u_2}^\dagger a_{u_3} a_{u_4} \rangle_{\Psi_\alpha} \right|, \quad (9.5)$$

We are going to prove:

$$\sum_{u \in P_L} \left| \langle a_0^\dagger a_0^\dagger a_u a_{-u} \rangle_{\Psi_\alpha} \right| = 0 \quad (9.6)$$

$$\sum_{u_2, u_3, u_4 \in P_L} \left| \langle a_0^\dagger a_{u_2}^\dagger a_{u_3} a_{u_4} \rangle_{\Psi_\alpha} \right| \leq \Lambda^2 \varrho^{3-5\eta} \quad (9.7)$$

$$\sum_{u_i \in P_L \text{ and } u_1 \neq u_3, u_4} \left| \langle a_{u_1}^\dagger a_{u_2}^\dagger a_{u_3} a_{u_4} \rangle_{\Psi_\alpha} \right| \leq \Lambda^3 \varrho^{5-9\eta} \quad (9.8)$$

First we note (9.6) is trivial. Because if $\beta \in M_\alpha$, then $P_L(\beta, \alpha)$ is non-trivial, which tells if $\beta(u) < \alpha(u)$ then $\beta(-u) = \alpha(-u)$.

Then we prove (9.7) concerning $u_{2,3,4} \in P_L$. By definition of M_α , one can see that if $\langle \beta | a_0^\dagger a_{u_2}^\dagger a_{u_3} a_{u_4} | \gamma \rangle \neq 0$, then $u_3 \neq u_4$ and $\gamma \notin M_\alpha(u_2)$, i.e., $\gamma(u_2) < \alpha(u_2)$. Furthermore, with the definition of f_α (2.3), we have

$$f(\beta) = \sqrt{\frac{\alpha(u_3)\alpha(u_4)}{\beta(0)\alpha(u_2)}} f(\gamma) \quad (9.9)$$

Combine with Lemma 14, we obtain

$$\left| \left\langle a_0^\dagger a_{u_2}^\dagger a_{u_3} a_{u_4} \right\rangle_{\Psi_\alpha} \right| \leq \alpha(u_3)\alpha(u_4) \sum_{\gamma \notin M_\alpha(u_2)} |f(\gamma)|^2 \quad (9.10)$$

Applying (6.6) in Lemma 11, we obtain

$$\left| \left\langle a_0^\dagger a_{u_2}^\dagger a_{u_3} a_{u_4} \right\rangle_{\Psi_\alpha} \right| \leq \text{const.} \alpha(u_3)\alpha(u_4) \varrho^{1-4\eta}, \quad (9.11)$$

which implies (9.7).

At last, we prove (9.8). Similarly, we have

$$\left| \left\langle a_{u_1}^\dagger a_{u_2}^\dagger a_{u_3} a_{u_4} \right\rangle_{\Psi_\alpha} \right| \leq \alpha(u_3)\alpha(u_4) \sum_{\gamma \notin M_\alpha(u_1) \cup M_\alpha(u_2)} |f(\gamma)|^2 \quad (9.12)$$

Again, using Lemma 11, we obtain

$$\left| \left\langle a_{u_1}^\dagger a_{u_2}^\dagger a_{u_3} a_{u_4} \right\rangle_{\Psi_\alpha} \right| \leq \text{const.} \alpha(u_3)\alpha(u_4) \varrho^{2-8\eta}, \quad (9.13)$$

which implies (9.8). At last, combine (9.6)-(9.8) and we obtain

$$|\langle H_{\tilde{L}\tilde{L}} \rangle_{\Psi_\alpha}| \leq \varrho^{11/4} \Lambda \quad (9.14)$$

■

10 Proof of Lemma 9

We start the proof with estimating $\langle a_{u_1}^\dagger a_{u_2}^\dagger a_{k_1} a_{k_2} \rangle_{\Psi_\alpha}$ when $u_1 = \pm u_2 \in P_L$. By the definition of M_α , if $\beta \in M_\alpha$, $u \in P_L$ and $\beta(u) < \alpha(u)$, then $\beta(u) = \alpha(u) - 1$ and $\beta(-u) = \alpha(-u)$. So we have:

$$\langle a_{u_1}^\dagger a_{u_2}^\dagger a_{k_1} a_{k_2} \rangle_{\Psi_\alpha} = 0, \quad \text{for } \forall k_1, k_2 \in P_H, u_1 = \pm u_2 \in P_L \quad (10.1)$$

For the other cases, we leave the bounds in the following lemma.

Lemma 15. Recall $P_{\tilde{L}} = P_0 \cup P_L$. For $u, u_1, u_2 \in P_{\tilde{L}}$ and $k, k_1, k_2 \in P_H$, we have

$$\left| \sum_{u \in P_{\tilde{L}}}^{k \in P_H} V_{u-k} \langle a_u^\dagger a_{-u}^\dagger a_k a_{-k} \rangle_{\Psi_\alpha} + \alpha(0)^2 \|Vw\|_1 \right| \leq \varepsilon_4 N^2 \quad (10.2)$$

$$\left| \sum_{u_1, 2 \in P_{\tilde{L}}, u_1 \neq \pm u_2}^{k_{1,2} \in P_H} V_{u_1-k_1} \langle a_{u_1}^\dagger a_{u_2}^\dagger a_{k_1} a_{k_2} \rangle_{\Psi_\alpha} + \sum_{u_1, u_2 \in P_{\tilde{L}}}^{u_1 \neq \pm u_2} 2\alpha(u_1)\alpha(u_2) \|Vw\|_1 \right| \leq \varepsilon_5 N^2 \quad (10.3)$$

$$\sum_{u_1, u_2 \in P_{\tilde{L}}, k_1, k_2 \in P_H}^{u_1 \neq u_2} \left| \langle a_{u_1}^\dagger a_{k_1}^\dagger a_{u_2} a_{k_2} \rangle_{\Psi_\alpha} \right| \leq \varepsilon_6 N^2 \quad (10.4)$$

where $\varepsilon_4, \varepsilon_5, \varepsilon_6$ are independent of α and $\lim_{\varrho \rightarrow 0} \varepsilon_i = 0$ for $i = 4, 5, 6$.

Proof. Proof of Lemma 9

Combine the bounds in (10.1), (10.2), (10.3) and (10.4). ■

10.1 Proof of Lemma 15

Proof. First we prove (10.2) concerning $u \in P_{\tilde{L}}$ and $k \in P_H$. By the definition of M_α , if $\langle a_u^\dagger a_{-u}^\dagger a_k a_{-k} \rangle_{\Psi_\alpha} \neq 0$, then u must be zero. With the property of f_α in Lemma 5 (4.9), one can see that if $\langle \beta | a_0^\dagger a_{-0}^\dagger a_k a_{-k} | \gamma \rangle \neq 0$, then

$$f_\alpha(\gamma) = -w_k \sqrt{\gamma(0)^2 - \gamma(0)} \Lambda^{-1} f_\alpha(\beta) \quad (10.5)$$

Together with Lemma 14, we have

$$\langle a_0^\dagger a_0^\dagger a_k a_{-k} \rangle_{\Psi_\alpha} = -w_k \sum_{\beta \in M_\alpha, \mathcal{A}_{k, -k}^{0,0} \beta \in M_\alpha} (\beta(0)^2 - \beta(0)) \Lambda^{-1} |f_\alpha(\beta)|^2, \quad (10.6)$$

Recall the definitions of M_α^B 's in Def. 4. One can see if $\beta(0) > 1$, then $\beta \in M_\alpha, \mathcal{A}_{k, -k}^{0,0} \beta \in M_\alpha$ is equivalent to $\beta \in M_\alpha^B(k) \cap M_\alpha^B(-k)$. So, we have

$$\langle a_0^\dagger a_0^\dagger a_k a_{-k} \rangle_{\Psi_\alpha} = -w_k \sum_{\beta \in M_\alpha^B(k) \cap M_\alpha^B(-k)} (\beta(0)^2 - \beta(0)) \Lambda^{-1} |f_\alpha(\beta)|^2, \quad (10.7)$$

Using the bound on $\sum_{\beta \notin M_\alpha^B(k)} |f_\alpha(\beta)|^2$ (6.23) and the bounds on $Q_\alpha(0)$, $Q_\alpha(0,0)$ in (6.29) and (6.30). We obtain that

$$\left| \sum_{\beta \in M_\alpha^B(k) \cap M_\alpha^B(-k)} (\beta(0)^2 - \alpha(0)^2) |f_\alpha(\beta)|^2 - \alpha(0)^2 \right| \leq O(\varrho^{1/6} N^2) \quad (10.8)$$

Insert (10.8) into (10.7). Then summing up $k \in P_H$, with $u = 0$, we obtain

$$\begin{aligned} & \left| \sum_{u \in P_{\tilde{L}}} \sum_{k \in P_H} V_{u-k} \langle a_u^\dagger a_{-u}^\dagger a_k a_{-k} \rangle_{\Psi_\alpha} + \alpha(0)^2 \|Vw\|_1 \right| \\ & \leq \alpha(0)^2 \left| \sum_{k \in P_H} -V_k w_k \Lambda^{-1} + \|Vw\|_1 \right| + O(\varrho^{1/6-3\eta} N^2) \end{aligned} \quad (10.9)$$

Combine with the fact $\lim_{\varrho \rightarrow 0} \left| \sum_{k \in P_H} -V_k w_k \Lambda^{-1} + \|Vw\|_1 \right| = 0$, we obtain the desired result (10.2).

Next, we prove (10.3) concerning $u_1 \neq \pm u_2 \in P_{\tilde{L}}$ and $k_1, k_2 \in P_H$. Using the result 2 in Lemma 5, one can see

$$\langle a_{u_1}^\dagger a_{u_2}^\dagger a_{k_1} a_{k_2} \rangle_{\Psi_\alpha} = 0 \text{ when } u_2 \in B_L(u_1) \quad (10.10)$$

So from now on, we assume $u_2 \notin B_L(u_1)$. Using the property of f_α in Lemma 5, we have that if $\langle \beta | a_{u_1}^\dagger a_{u_2}^\dagger a_{k_1} a_{k_2} | \gamma \rangle \neq 0$ and $\beta, \gamma \in M_\alpha$, then

$$f(\gamma) = C_\beta \sqrt{-w_{k_1}} \sqrt{-w_{k_2}} \sqrt{\beta(u_1) \beta(u_2)} f(\beta) \quad (10.11)$$

Here C_β depends on β and C_β is no more than 2. Especially, when $\beta \in M_\alpha(-k_1) \cap M_\alpha(-k_2)$, $C_\beta = 2$. Again with Lemma 14, for fixed $u_1, u_2 \notin B_L(u_1)$, k_1 and k_2 , we have

$$\langle a_{u_1}^\dagger a_{u_2}^\dagger a_{k_1} a_{k_2} \rangle_{\Psi_\alpha} = \sqrt{-w_{k_1}} \sqrt{-w_{k_2}} \sum_{\beta \in M_\alpha, \mathcal{A}_{k_1, k_2}^{u_1, u_2} \beta \in M_\alpha} C_\beta \beta(u_1) \beta(u_2) |f(\beta)|^2, \quad (10.12)$$

First, using the facts $|k_1 + k_2| \leq 2\varrho^{1/3} \eta_L$ and the bound on dw_p/dp (2.14), we obtain $|w_{k_1} - w_{k_2}| \leq \varrho^{1/4}$, then

$$\left| \sqrt{-w_{k_1}} \sqrt{-w_{k_2}} + w_{k_1} \right| \leq \varrho^{1/4} \quad (10.13)$$

Insert (10.13) into (10.12), we have

$$\langle a_{u_1}^\dagger a_{u_2}^\dagger a_{k_1} a_{k_2} \rangle_{\Psi_\alpha} = (-w_{k_1} + O(\varrho^{1/4})) \sum_{\beta \in M_\alpha, \mathcal{A}_{k_1, k_2}^{u_1, u_2} \beta \in M_\alpha} C_\beta \beta(u_1) \beta(u_2) |f_\alpha(\beta)|^2. \quad (10.14)$$

Now we bound

$$\sum_{\beta \in M_\alpha, \mathcal{A}_{k_1, k_2}^{u_1, u_2} \beta \in M_\alpha} C_\beta \beta(u_1) \beta(u_2) |f_\alpha(\beta)|^2.$$

In the case $\beta \notin M_\alpha(-k_1) \cap M_\alpha(-k_2)$, using the result in (6.5) and $|C_\beta| \leq 2$, we have

$$\left| \sum_{\beta \notin M_\alpha(k_1) \cap M_\alpha(k_2)} C_\beta \beta(u_1) \beta(u_2) |f_\alpha(\beta)|^2 \right| \leq \varrho \alpha(u_1) \alpha(u_2) \quad (10.15)$$

In the case $\beta \in M_\alpha(-k_1) \cap M_\alpha(-k_2)$, we have $C_\beta = 2$. Recall $M_\alpha^{u_1, u_2}$ in (6.25) and (6.27). By the definition of M_α (Def. 5 rule 4), we have

$$\sum_{\beta \in \cap_{i=1}^2 M_\alpha(-k_i), \mathcal{A}_{k_1, k_2}^{u_1, u_2} \beta \in M_\alpha} C_\beta \beta(u_1) \beta(u_2) |f_\alpha(\beta)|^2 = \sum_{\beta \in A} 2\beta(u_1) \beta(u_2) |f_\alpha(\beta)|^2, \quad (10.16)$$

where A is the following subset of M_α : $(p_1, p_2, p_3, p_4 := u_1, u_2, k_1, k_2)$

$$A = (\cap_{i=1}^4 M_\alpha^B(p_i)) \cap (\cap_{i=1}^2 M_\alpha(-k_i)) \cap M_\alpha^{u_1, u_2} \quad (10.17)$$

Using the results in Lemma 11 and Lemma 13((6.5), (6.23), (6.24), (6.26) and (6.28)) and $\alpha(u) \leq m_c$ for $u \in P_L$, we obtain that if $u_1, u_2 \in P_L$

$$\left| \sum_{\beta \in A} \beta(u_1) \beta(u_2) |f_\alpha(\beta)|^2 - \alpha(u_1) \alpha(u_2) \right| \leq O(\varrho^{1/6-6\eta}) \quad (10.18)$$

and if $u_1 = 0, u_2 \in P_L$, we have

$$\left| \sum_{\beta \in A} \beta(u_1) \beta(u_2) f_\alpha^2(\beta) - \alpha(u_1) \alpha(u_2) \right| \leq O(\varrho^{1/6-3\eta} N) \quad (10.19)$$

Inserting (10.15), (10.18) and (10.19) into (10.14), with the fact $|w_p| \leq 4\pi a|p|^{-2}$, we obtain that for $u_1, u_2 \in P_L$:

$$\left| \langle a_{u_1}^\dagger a_{u_2}^\dagger a_{k_1} a_{k_2} \rangle_{\Psi_\alpha} + 2w_{k_1} \alpha(u_1) \alpha(u_2) \right| \leq O(\varrho^{1/6-8\eta}) \quad (10.20)$$

and for $u_1 = 0, u_2 \in P_L$,

$$\left| \langle a_{u_1}^\dagger a_{u_2}^\dagger a_{k_1} a_{k_2} \rangle_{\Psi_\alpha} + 2w_{k_1} \alpha(u_1) \alpha(u_2) \right| \leq O(\varrho^{1/6-5\eta} N). \quad (10.21)$$

Furthermore, the smoothness of V implies $|V_{u_1-k_1}-V_{k_1}| \leq \varrho^{1/4}$. So summing up $u_1, u_2 \notin B_L(u_1)$ and k_1, k_2 , we obtain

$$\begin{aligned}
& \left| \sum_{u_1, u_2, k_1, k_2} V_{u_1-k_1} \langle a_{u_1}^\dagger a_{u_2}^\dagger a_{k_1} a_{k_2} \rangle_{\Psi_\alpha} + 2 \sum_{u_1, u_2 \in P_L \cup P_0}^{u_1 \neq \pm u_2} \alpha(u_1) \alpha(u_2) \|Vw\|_1 \right| \\
& \leq 2 \sum_{u_1, u_2 \in P_L \cup P_0}^{u_1 \neq \pm u_2} \alpha(u_1) \alpha(u_2) \left| \sum_{k_1} |V_{k_1} w_{k_1}| \Lambda^{-1} - \|Vw\|_1 \right| + O(\varrho^{1/6-17\eta} N^2) \\
& + \text{const.} \sum_{u_1, u_2: u_2 \in B_L(u_1)} \alpha(u_1) \alpha(u_2) \tag{10.22}
\end{aligned}$$

One can see the first line of the right side is less than $\varepsilon_5 N^2/2$. Here ε_5 is independent of α and $\lim_{\varrho} \varepsilon_5 = 0$. With the bound $\alpha(u) \leq m_c$ for $u \in P_L$, we can obtain that the second line of the right side is also less than $\varepsilon_5 N^2/2$. So we arrive at the desired result (10.3).

At last, we prove (10.4) concerning $u_{1,2} \in P_L$, $u_1 \neq u_2$ and $k_{1,2} \in P_H$. By definition of M_α and f_α , if $\langle \beta | a_{u_1}^\dagger a_{k_1}^\dagger a_{u_2} a_{k_2} | \gamma \rangle \neq 0$ and $\beta, \gamma \in M_\alpha$, then $\gamma \notin M_\alpha(u_1) \cup M_\alpha(k_2)$, $\beta \notin M_\alpha(u_2) \cup M_\alpha(k_1)$ and

$$|f_\alpha(\gamma)| \leq \text{const.} \left| \sqrt{\frac{\alpha(u_1)}{\alpha(u_2)}} \sqrt{\frac{w_{k_2}}{w_{k_1}}} \right| |f_\alpha(\beta)| \tag{10.23}$$

This implies

$$\left| f_\alpha(\beta) f_\alpha(\gamma) \langle \beta | a_{u_1}^\dagger a_{k_1}^\dagger a_{u_2} a_{k_2} | \gamma \rangle \right| \leq \text{const.} \alpha(u_1) \left| \sqrt{\frac{w_{k_2}}{w_{k_1}}} \right| |f_\alpha(\beta)|^2 \tag{10.24}$$

Summing up $\beta \notin M_\alpha(u_2) \cup M_\alpha(k_1)$, with the bound on $\sum_\beta |f_\alpha(\beta)|^2$ (6.8), we have

$$\begin{aligned}
\left| \langle a_{u_1}^\dagger a_{k_1}^\dagger a_{u_2} a_{k_2} \rangle_{\Psi_\alpha} \right| & \leq \text{const.} \alpha(u_1) \left| \sqrt{\frac{w_{k_2}}{w_{k_1}}} \right| \sum_{\beta \notin M_\alpha(u_2) \cup M_\alpha(k_1)} |f_\alpha(\beta)|^2 \\
& \leq \alpha(u_1) |\sqrt{w_{k_1} w_{k_2}}| \varrho^{3-8\eta} \tag{10.25}
\end{aligned}$$

At last, using $|w_p| \leq 4\pi a|p|^{-2}$ and $|k_1| \sim |k_2|$, we have

$$\begin{aligned}
\sum_{u_1, u_2 \in P_L, k_1, k_2 \in P_H} \left| \langle a_{u_1}^\dagger a_{k_1}^\dagger a_{u_2} a_{k_2} \rangle_{\Psi_\alpha} \right| & \leq \sum_{u_1} \alpha(u_1) \varrho^{3-9\eta} \Lambda \\
& \leq \Lambda^2 \varrho^{5/2} \tag{10.26}
\end{aligned}$$

■

11 Proof of Lemma 10

In this section, we will prove Lemma 10 involving interaction energy between particles with momenta in P_H . We will show that the only contribution to the accuracy we need comes from four high momentum particles, to be computed in Lemma 16 (11.3). We start with separating $\langle H_{HH} \rangle_{\Psi_\alpha}$ into the main terms and the error terms.

Define $M_\alpha(k_1, k_2, k_3, k_4, u_1, u_2) \subset M_\alpha \otimes M_\alpha$ as the set of (β, γ) 's where β and γ can be created from the same $\tilde{\alpha} \in M_\alpha$ as follows,

$$\begin{aligned} & M_\alpha(k_1, k_2, k_3, k_4, u_1, u_2) \\ \equiv & \{(\beta, \gamma) \in M_\alpha \otimes M_\alpha : \exists \tilde{\alpha} \in M_\alpha \text{ s.t. } \mathcal{A}_{k_1, k_2}^{u_1, u_2} \tilde{\alpha} = \beta \text{ and } A_{k_3, k_4}^{u_1, u_2} \tilde{\alpha} = \gamma\}, \end{aligned} \quad (11.1)$$

where $k_1, k_2, k_3, k_4 \in P_H$ and $u_1, u_2 \in P_{\tilde{L}}$. For $k_1, k_2, k_3, k_4 \in P_H$, $u_1, u_2 \in P_{\tilde{L}}$, we define $A_{u_1, u_2, k_1, k_2, k_3, k_4}$ as

$$A_{u_1, u_2, k_1, k_2, k_3, k_4} \equiv \sum_{(\beta, \gamma) \in M_\alpha(k_1, k_2, k_3, k_4, u_1, u_2)} f_\alpha(\beta) \overline{f_\alpha(\gamma)} \left\langle \beta | a_{k_1}^\dagger a_{k_2}^\dagger a_{k_3} a_{k_4} | \gamma \right\rangle \quad (11.2)$$

With (11.2), we can separate the expectation value of H_{HH} into two parts, main term (Lemma 16) and error term (Lemma 17).

Lemma 16. *Summing up $k_1, k_2, k_3, k_4 \in P_H$, $k_i \neq k_j$ for $i \neq j$, $u_1, u_2 \in P_{\tilde{L}}$, we have*

$$\left| \sum_{u_i, k_i} V_{k_1 - k_3} \Lambda^{-1} A_{u_1, u_2, k_1, k_2, k_3, k_4} - N_\alpha |\Lambda|^{-1} \|V w^2\|_1 \right| \leq \frac{\varepsilon_3}{2} \varrho^2 \Lambda, \quad (11.3)$$

where ε_3 is independent of α and $\lim_{\varrho \rightarrow 0} \varepsilon_3 = 0$.

Lemma 17. *Define $M_\alpha(k_1, k_2, k_3, k_4)$ as follows,*

$$M_\alpha(k_1, k_2, k_3, k_4) \equiv \cup_{u_1, u_2 \in P_{\tilde{L}}} M_\alpha(k_1, k_2, k_3, k_4, u_1, u_2). \quad (11.4)$$

Then we have

$$\sum_{k_i \in P_H} \sum_{(\beta, \gamma) \notin M_\alpha(k_1, k_2, k_3, k_4)} V_0 \Lambda^{-1} \left| f_\alpha(\beta) \overline{f_\alpha(\gamma)} \left\langle \beta | a_{k_1}^\dagger a_{k_2}^\dagger a_{k_3} a_{k_4} | \gamma \right\rangle \right| \leq \frac{\varepsilon_3}{2} \varrho^2 \Lambda \quad (11.5)$$

Here $k_i \neq k_j$ for $i \neq j$.

11.1 Proof of Lemma 10

Proof. By the definition of M_α , we have $\beta(k) \leq 1$ when $k \in P_H$ and $\beta \in M_\alpha$. So the expectation value of $a_{k_1}^\dagger a_{k_2}^\dagger a_{k_3} a_{k_4}$ must be zero when $k_1 = k_2$ or $k_3 = k_4$. Together with the definition of H_{HH} , we can rewrite $\langle H_{HH} \rangle_{\Psi_\alpha}$ as

$$\langle H_{HH} \rangle_{\Psi_\alpha} = \sum_{k_i \in P_H} \sum_{\substack{k_i \neq k_j \\ \beta, \gamma \in M_\alpha}} V_{k_1-k_3} \Lambda^{-1} f_\alpha(\beta) \overline{f_\alpha(\gamma)} \langle \beta | a_{k_1}^\dagger a_{k_2}^\dagger a_{k_3} a_{k_4} | \gamma \rangle \quad (11.6)$$

On the other hand, if $\beta, \gamma \in M_\alpha$ and $\langle \beta | a_{k_1}^\dagger a_{k_2}^\dagger a_{k_3} a_{k_4} | \gamma \rangle \neq 0$ for some $k_{1,2,3,4} \in P_H$, then by the fact $P_L(\beta, \alpha) = P_L(\gamma, \alpha)$ is non-trivial (Def. 5), there **at most** one pair of $\{u_1, u_2\}$ satisfying

$$(\beta, \gamma) \in M_\alpha(k_1, k_2, k_3, k_4, u_1, u_2) \quad (11.7)$$

So combine (11.3) and (11.5), with $|V_{k_1-k_3}| \leq V_0$, we obtain the desired result (5.13). ■

11.2 Proof of Lemma 16

Proof. We start with giving the bounds on $A_{u_1, u_2, k_1, k_2, k_3, k_4}$.

Lemma 18. *When $u_1, u_2 \in P_L$ and $u_1 = \pm u_2$ or $u_2 \in B_L(u_1)$, for any $k_i \in P_H$, we have*

$$A_{u_1, u_2, k_1, k_2, k_3, k_4} = 0 \quad (11.8)$$

In other cases, $A_{u_1, u_2, k_1, k_2, k_3, k_4}$ is bounded by $(P_0 = \{0\})$

$$\begin{aligned} & |A_{u_1, u_2, k_1, k_2, k_3, k_4} - \alpha(u_1) \alpha(u_2) F_a(u_1, u_2)^2 w_{k_1} w_{k_3} \Lambda^{-2}| \quad (11.9) \\ & \leq \begin{cases} \varrho^{1/8} \Lambda^{-2} \alpha(u_1) \alpha(u_2), & u_1, u_2 \in P_L \\ \varrho^{1/8} \Lambda^{-2} N \alpha(u_2), & u_1 \in P_0, u_2 \in P_L \\ \varrho^{1/8} \Lambda^{-2} N^2, & u_1 = u_2 \in P_0, \end{cases} \end{aligned}$$

where $F_a(u_1, u_2) = 1$ when $u_1 = u_2 = 0$, otherwise $F_a(u_1, u_2) = 2$.

Proof. Proof of Lemma 18

First we prove (11.8). One can see that it follows the definition of $A_{u_1, u_2, k_1, k_2, k_3, k_4}$ and the result 2 in Lemma 5.

Then we prove (11.9) when $u_1, u_2 \in P_L$. If (11.7) holds, by the definition of M_α ($k_1, k_2, k_3, k_4, u_1, u_2$) in (11.1), we can see that there exists

$\tilde{\alpha} \in M_\alpha$, $\mathcal{A}_{k_1, k_2}^{u_1, u_2} \tilde{\alpha} = \beta$ and $\mathcal{A}_{k_3, k_4}^{u_1, u_2} \tilde{\alpha} = \gamma$. With definition of f_α , when $\tilde{\alpha} \in \cap_{i=1}^4 M_\alpha(-k_i)$, we have

$$\begin{aligned} f_\alpha(\beta) &= -F_a(u_1, u_2) \sqrt{\alpha(u_1)\alpha(u_2)} \Lambda^{-1} \sqrt{-w_{k_1}} \sqrt{-w_{k_2}} f_\alpha(\tilde{\alpha}) \\ f_\alpha(\gamma) &= -F_a(u_1, u_2) \sqrt{\alpha(u_1)\alpha(u_2)} \Lambda^{-1} \sqrt{-w_{k_1}} \sqrt{-w_{k_2}} f_\alpha(\tilde{\alpha}). \end{aligned} \quad (11.10)$$

And when $\tilde{\alpha} \notin \cap_{i=1}^4 M_\alpha(-k_i)$, we have the following bound on $|f_\alpha(\beta)f_\alpha(\gamma)|$,

$$|f_\alpha(\beta)f_\alpha(\gamma)| \leq 4\alpha(u_1)\alpha(u_2)\Lambda^{-2} \prod_{i=1}^4 |\sqrt{w_{k_i}}| |f_\alpha(\tilde{\alpha})|^2 \quad (11.11)$$

On the other hand, if $k_i \in P_H$ for $1 \leq i \leq 4$ and

$$\beta, \gamma \in M_\alpha \text{ and } \langle \beta | a_{k_1}^\dagger a_{k_2}^\dagger a_{k_3} a_{k_4} | \gamma \rangle \neq 0, \quad (11.12)$$

then by the definition of M_α , we have $\beta(k_1) = \beta(k_2) = 1$ and $\gamma(k_3) = \gamma(k_4) = 1$. This implies

$$\langle \beta | a_{k_1}^\dagger a_{k_2}^\dagger a_{k_3} a_{k_4} | \gamma \rangle = 1 \quad (11.13)$$

Combine (11.10), (11.11) and (11.13), we obtain that if $\mathcal{A}_{k_1, k_2}^{u_1, u_2} \tilde{\alpha} = \beta$ and $\mathcal{A}_{k_3, k_4}^{u_1, u_2} \tilde{\alpha} = \gamma$, when $\tilde{\alpha} \in \cap_{i=1}^4 M_\alpha(-k_i)$, then

$$f_\alpha(\beta) \overline{f_\alpha(\gamma)} \langle \beta | a_{k_1}^\dagger a_{k_2}^\dagger a_{k_3} a_{k_4} | \gamma \rangle = F_a(u_1, u_2)^2 \tilde{\alpha}(u_1) \tilde{\alpha}(u_2) \Lambda^{-2} \prod_{i=1}^4 \sqrt{-w_{k_i}} |f_\alpha(\tilde{\alpha})|^2 \quad (11.14)$$

When $\tilde{\alpha} \notin \cap_{i=1}^4 M_\alpha(-k_i)$, using (6.5), we have

$$\sum_{\tilde{\alpha} \notin \cap_{i=1}^4 M_\alpha(-k_i)} \left| f_\alpha(\beta) \overline{f_\alpha(\gamma)} \langle \beta | a_{k_1}^\dagger a_{k_2}^\dagger a_{k_3} a_{k_4} | \gamma \rangle \right| \leq \text{const. } \varrho^{3/2} \alpha(u_1) \alpha(u_2) \Lambda^{-2} \quad (11.15)$$

So with (11.14) and (11.15), we can see

$$\begin{aligned} & A_{u_1, u_2, k_1, k_2, k_3, k_4} + O(\varrho^{3/2}) \alpha(u_1) \alpha(u_2) \Lambda^{-2} \\ &= F_a(u_1, u_2)^2 \Lambda^{-2} \prod_{i=1}^4 \sqrt{-w_{k_i}} \sum_{\tilde{\alpha} \in A} \tilde{\alpha}(u_1) \tilde{\alpha}(u_2) |f(\tilde{\alpha})|^2 \end{aligned} \quad (11.16)$$

Where A is defined as the set

$$A \equiv \{ \tilde{\alpha} \in M_\alpha : \mathcal{A}_{k_1, k_2}^{u_1, u_2} \tilde{\alpha} = \beta \in M_\alpha, \mathcal{A}_{k_3, k_4}^{u_1, u_2} \tilde{\alpha} = \gamma \in M_\alpha, \tilde{\alpha} \in \cap_{i=1}^4 M_\alpha(-k_i) \}$$

With the definition of M_α , denoting u_i, k_j 's as p_1, p_2, \dots, p_6 , we obtain

$$A = \cap_{i=1}^6 M_\alpha^B(p_i) \cap \cap_{i=1}^4 M_\alpha(-k_i) \cap M_\alpha^{u_1, u_2} \quad (11.17)$$

Here $M_\alpha^{u_1, u_2}$ is defined in Lemma 13 (6.25). Using the results in Lemma 13, we have $\sum_{\tilde{\alpha} \in A} |f(\tilde{\alpha})|^2$ bounded by

$$1 \leq \sum_{\tilde{\alpha} \in A} |f(\tilde{\alpha})|^2 \leq 1 - O(\varrho^{1/6}) \quad (11.18)$$

On the other hand, using (10.13), with the fact $|k_1 + k_2| = |k_3 + k_4| \leq \varrho^{1/3} \varrho^{-\eta}$, one can bound the $\prod_{i=1}^4 \sqrt{-w_{k_i}}$ in (11.16) as follows

$$\left| \prod_{i=1}^4 \sqrt{-w_{k_i}} - w_{k_1} w_{k_3} \right| \leq O(\varrho^{1/4-\eta}) \quad (11.19)$$

Inserting (11.19) and (11.18) into (11.16), with the fact $\tilde{\alpha} \in M_\alpha(u_i)$, i.e. $\tilde{\alpha}(u_i) = \alpha(u_i)$ ($i = 1, 2$), we arrive at the desired result (11.9).

Similarly, using the bounds on $Q_\alpha(0)$ and $Q_\alpha(0, 0)$ in (6.29) and (6.30), one can prove (11.9) when one of u_i belongs to P_0 or both of them belong to P_0 . \blacksquare

With (11.9), summing up k_1, k_3, u_1, u_2 , one can easily obtain the desired result (11.3). \blacksquare

11.3 Proof of Lemma 17

Proof. As in [16], to estimate the error term of the interaction of particles with high momenta, we need to use a new tool. We start with defining the set $M_\alpha(\tilde{\alpha}, s, \{v_1, \dots, v_t\})$. Let $v_1, \dots, v_t \in P_L$ and being in different small boxes B_L , i.e.,

$$B_L(v_i) \neq B_L(v_j), \text{ for } i \neq j. \quad (11.20)$$

For non-negative integers s, t satisfying $s + t \in 2\mathbb{N}$ and $\tilde{\alpha} \in M_\alpha$, define

$$M(\tilde{\alpha}, s, \{v_1, \dots, v_t\}) \equiv \left\{ \beta \in M_\alpha : \beta = \prod_{i=m+1}^{(s+t)/2} \mathcal{A}_{p_{2i-1}, p_{2i}}^{u_{2i-1}, u_{2i}} \prod_{i=1}^m \mathcal{A}_{p_{2i-1}, p_{2i}}^{u_{2i-1}, u_{2i}} \tilde{\alpha} \right\} \quad (11.21)$$

where the u_i 's and k_i 's satisfy

1. $u_i \in P_L$, $p_i \in P_H$, $1 \leq i \leq s + t$. And $u_i = 0$ for $i \leq 2m$.

2. $\{u_i, 1 \leq i \leq s+t\}$ is a permutation of s zeros and $\{v_1, \dots, v_t\}$
3. for any fixed $2m+1 \leq j \leq s+t$, $\tilde{\alpha} \in M_\alpha(-p_j)$, i.e., $\tilde{\alpha}(-p_j) = \alpha(-p_j)$.
And $p_j \neq -p_i$ for $1 \leq i \leq s+t$.

. We note: if $\beta, \tilde{\alpha} \in M_\alpha$ have the form in (11.21), then it is easy to check

$$\prod_{i \in A} \mathcal{A}_{k_{2i-1}, k_{2i}}^{u_{2i-1}, u_{2i}} \tilde{\alpha} \in M_\alpha, \text{ for } A \subset \{1, \dots, (s+t)/2\} \quad (11.22)$$

By this definition, if (11.12) holds, then $\beta(u) = \gamma(u)$ for any $u \in P_{\tilde{L}}$, so there at least exists one $M_\alpha(\tilde{\alpha}, s, \{v_i, 1 \leq i \leq t\})$ satisfying

$$\beta \text{ and } \gamma \in M_\alpha(\tilde{\alpha}, s, \{v_1, \dots, v_t\}) \quad (11.23)$$

E.g. Using Lemma 4, we can see (11.23) holds when we choose $\tilde{\alpha} = \alpha$, $\{v_1, \dots, v_t\} = P_L(\beta, \alpha) = P_L(\gamma, \alpha)$.

Then, for any $M_\alpha(\tilde{\alpha}, s, \{v_1, \dots, v_t\})$, we define $N(\tilde{\alpha}, s, \{v_1, \dots, v_t\})$ as the set of the pairs (β, γ) such that

1. $\beta, \gamma \in M_\alpha(\tilde{\alpha}, s, \{v_1, \dots, v_t\})$
2. there exist $k_i, 1 \leq i \leq 4$ satisfying (11.12) but

$$(\beta, \gamma) \notin M_\alpha(k_1, k_2, k_3, k_4). \quad (11.24)$$

Here $M_\alpha(k_1, k_2, k_3, k_4)$ is defined in (11.4)

3. for any other $\tilde{\alpha}', s', \{v'_1, \dots, v'_{t'}\}$, if $\beta, \gamma \in M_\alpha(\tilde{\alpha}', s', \{v'_1, \dots, v'_{t'}\})$, then

$$s+t \leq s'+t' \quad (11.25)$$

We assume (11.23) and (11.12) holds. Clearly, $s+t=2$ or $t=0$ implies that $(\beta, \gamma) \in M_\alpha(k_1, k_2, k_3, k_4)$. Hence if $N(\tilde{\alpha}, s, \{v_1, \dots, v_t\})$ is not an empty set then

$$s+t \geq 4, \text{ and } t \geq 1 \quad (11.26)$$

By definition of $N(\tilde{\alpha}, s, \{v_1, \dots, v_t\})$ and (11.13), we can bound the left side of (11.5) as follows ($k_i \neq k_j$ for $i \neq j$)

$$\begin{aligned} & \sum_{k_i \in P_H} \sum_{\beta, \gamma \notin M_\alpha(k_1, k_2, k_3, k_4)} V_0 \Lambda^{-1} \left| f_\alpha(\beta) \overline{f_\alpha(\gamma)} \left\langle \beta | a_{k_1}^\dagger a_{k_2}^\dagger a_{k_3} a_{k_4} | \gamma \right\rangle \right| \quad (11.27) \\ & \leq \sum_{\tilde{\alpha}, s, \{v_1, \dots, v_t\}} V_0 \Lambda^{-1} |N(\tilde{\alpha}, s, \{v_1, \dots, v_t\})| \max_{\beta, \gamma \in M_\alpha(\tilde{\alpha}, s, \{v_1, \dots, v_t\})} \left| f_\alpha(\beta) \overline{f_\alpha(\gamma)} \right|, \end{aligned}$$

where $|N(\tilde{\alpha}, s, \{v_1, \dots, v_t\})|$ is the number of elements in this set. By definition of f_α and $\alpha(u) \leq m_c = \varrho^{-3\eta}$ for any $u \in P_L$, we have: if $\beta, \gamma \in M_\alpha(\tilde{\alpha}, s, \{v_1, \dots, v_t\})$ then

$$|f_\alpha(\beta)\overline{f_\alpha(\gamma)}| \leq \text{const.}^{t+s} \left| \frac{\alpha(0)}{|\Lambda|} \right|^s \left| \frac{\varrho^{-3\eta}}{|\Lambda|} \right|^t \max_{k \in P_H} \{|w_k|\}^{s+t} |f_\alpha(\tilde{\alpha})|^2$$

Then with the fact $|w_p| \leq 4\pi a|p|^{-2}$ and $\alpha(0) \leq N$, we obtain

$$|f_\alpha(\beta)\overline{f_\alpha(\gamma)}| \leq \text{const.}^{t+s} (\varrho^{1-2\eta})^s (\varrho^{-5\eta})^t |\Lambda|^{-t} |f_\alpha(\tilde{\alpha})|^2 \quad (11.28)$$

So, the right hand side of (11.27) is bounded by

$$(11.27) \leq \sum_{\tilde{\alpha}, s, \{v_1 \dots v_t\}} |N(\tilde{\alpha}, s, \{v_1, \dots, v_t\})| \varrho^s (\varrho^{-6\eta})^{t+s} |\Lambda|^{-t-1} |f(\tilde{\alpha})|^2 \quad (11.29)$$

Define $N(\tilde{\alpha}, s, t)$ and $N(s, t)$ by

$$N(\tilde{\alpha}, s, t) \equiv \max_{\{v_1, \dots, v_t\}} \{|N(\tilde{\alpha}, s, \{v_1, \dots, v_t\})|\} \quad (11.30)$$

$$N(s, t) \equiv \max_{\tilde{\alpha}} \{N(\tilde{\alpha}, s, t)\} \quad (11.31)$$

With $N(\tilde{\alpha}, s, t)$ and $N(s, t)$, we can bound (11.29) by

$$\begin{aligned} (11.27) \leq (11.29) &\leq \sum_{\tilde{\alpha}, s, t} |f(\tilde{\alpha})|^2 \sum_{\{v_1 \dots v_t\}} N(\tilde{\alpha}, s, t) \varrho^s (\varrho^{-6\eta})^{t+s} |\Lambda|^{-t-1} \\ &\leq \sum_{s, t} \sum_{\{v_1 \dots v_t\}} N(s, t) \varrho^s (\varrho^{-6\eta})^{t+s} |\Lambda|^{-t-1} \end{aligned} \quad (11.32)$$

For fixed t , the total number of sets $\{v_1 \dots v_t, v_i \in P_L\}$ is bounded by

$$\sum_{\{v_1 \dots v_t\}} 1 \leq (\Lambda \varrho \eta_L^{-3})^t (t!)^{-1} \leq (\varrho^{1-3\eta})^t |\Lambda|^t (t!)^{-1}$$

On the other hand t is bounded above by the number of B_L 's in P_L , i.e.,

$$t \leq |P_L| / \max_i \{|B_L^i|\} \leq \text{const.} \varrho^{1-3\eta-3\kappa_L},$$

where $|P_L|$ and $|B_L^i|$ are the volumes of P_L and the small box B_L^i 's. Combine with (11.26), we have

$$(11.27) \leq \varrho^{1-4\eta-3\kappa_L} \sum_{t=1} \sum_{s: s+t \geq 4} N(s, t) (\varrho^{1-9\eta})^{s+t} |\Lambda|^{-1} (t!)^{-1} \quad (11.33)$$

We claim that $N(s, t)$ is bounded with the following lemma, which will be proved in next subsection.

Lemma 19. For any $N(\alpha, s, \{v_1, \dots, v_t\})$, $s + t \geq 4$ and $t \geq 1$, we have

$$|N(\alpha, s, \{v_1, \dots, v_t\})| \leq t! t^{(\frac{3t}{4})} |\Lambda|^{\frac{s+t}{4}+1} (\varrho^{-\eta})^{t+s} \quad (11.34)$$

Together with (11.33), we obtain

$$\begin{aligned} (11.27) &\leq \sum_{t=1}^{\varrho^{1-4\eta-3\kappa_L}} \sum_{s:s+t \geq 4} (\varrho^{1-10\eta})^{s+t} t^{(\frac{3t}{4})} |\Lambda|^{\frac{s+t}{4}} \\ &= \sum_{t=1}^{\varrho^{1-4\eta-3\kappa_L}} \sum_{s:s+t \geq 4} (\varrho^{1-10\eta} \Lambda^{1/4})^s (\varrho^{1-10\eta} t^{3/4} \Lambda^{1/4})^t \end{aligned} \quad (11.35)$$

Then $\kappa_L \leq 1/2$ implies $\varrho^{1-10\eta} \Lambda^{1/4} < 1$ and $\varrho^{1-10\eta} t^{3/4} \Lambda^{1/4} < 1$, therefore, we arrive at the desired result

$$(11.27) \leq O(1) \ll \varrho^2 \Lambda \quad (11.36)$$

■

11.4 Proof of Lemma 19

We now prove Lemma 19.

Proof. Since $(\beta, \gamma) \in N(\tilde{\alpha}, s, \{v_1, \dots, v_t\})$, we can express them as (11.21):

$$\beta = \prod_{i=1}^{(s+t)/2} \mathcal{A}_{q_{2i-1}, q_{2i}}^{u_{2i-1}, u_{2i}} \tilde{\alpha}, \quad \gamma = \prod_{i=1}^{(s+t)/2} \mathcal{A}_{\tilde{q}_{2i-1}, \tilde{q}_{2i}}^{\tilde{u}_{2i-1}, \tilde{u}_{2i}} \tilde{\alpha} \quad (11.37)$$

We note that for any $1 \leq i \leq (s+t)/2$

$$\{q_{2i-1}, q_{2i}\} \neq \{k_1, k_2\} \quad \text{and} \quad \{\tilde{q}_{2i-1}, \tilde{q}_{2i}\} \neq \{k_3, k_4\}, \quad (11.38)$$

otherwise one can see $(\beta, \gamma) \in M_\alpha(k_1, k_2, k_3, k_4)$, which contradicts with $(\beta, \gamma) \in N(\tilde{\alpha}, s, \{v_1, \dots, v_t\})$.

From (11.12), the sets $\{q_1, \dots, q_{2s+2t}\}$ and $\{\tilde{q}_1, \dots, \tilde{q}_{2s+2t}\}$ are very close, i.e.,

$$\{q_1, \dots, q_{2s+2t}\} - \{k_1, k_2\} = \{\tilde{q}_1, \dots, \tilde{q}_{2s+2t}\} - \{k_3, k_4\} \quad (11.39)$$

Denote the common elements in $\{q_i\}$ and $\{\tilde{q}_i\}$ by $p_1, p_2, \dots, p_{s+t-2}$. Then we have

$$\{q_i\} = k_1, k_2, p_1, p_2, \dots, p_{s+t-2}, \quad (11.40)$$

$$\{\tilde{q}_i\} = k_3, k_4, p_1, p_2, \dots, p_{s+t-2}, \quad (11.41)$$

We now construct a graph with vertices $\{k_1, k_2, k_3, k_4, p_i, 1 \leq i \leq s+t-2\}$. The edges of the graphs consisting of β edges $(q_{2i-1}, q_{2i}), 1 \leq i \leq (s+t)/2$ and γ edges $(\tilde{q}_{2j-1}, \tilde{q}_{2j}), 1 \leq j \leq (s+t)/2$. From (11.12), we know each $k_i (1 \leq i \leq 4)$ touches one edge and each $p_i (1 \leq i \leq s+t-2)$ touches two edges. So the graph can be decomposed into two chains and loops. Thus there exist $l, m_i \in \mathbb{Z}$ and $0 < m_1 < m_2 < \dots < m_l = s+t$ such that

$$\begin{aligned} \text{chains} \left\{ \begin{array}{l} k_1 \longleftrightarrow p_1 \longleftrightarrow p_2 \longleftrightarrow p_3 \cdots p_{2m_1-1} \longleftrightarrow k_2 \text{ (or } k_4) \\ k_3 \longleftrightarrow p_{2m_1} \longleftrightarrow p_{2m_1+1} \cdots p_{2m_2-2} \longleftrightarrow k_4 \text{ (or } k_2) \end{array} \right. \quad (11.42) \\ \text{loops} \left\{ \begin{array}{l} p_{2m_2-1} \longleftrightarrow p_{2m_2} \longleftrightarrow p_{2m_2+1} \cdots p_{2(m_3)-2} \longleftrightarrow p_{2m_2-1} \\ \dots\dots\dots \\ p_{2m_{l-1}-1} \longleftrightarrow p_{2m_{l-1}} \longleftrightarrow p_{2m_{l-1}+1} \cdots p_{2(m_l)-2} \longleftrightarrow p_{2m_{l-1}-1} \end{array} \right. \end{aligned}$$

Here we have relabeled the indices of p and do not distinguish β edges and γ edges. We also disregard the obvious symmetry $k_1 \rightarrow k_2$ and $k_3 \rightarrow k_4$. Due to the condition (11.25) and the facts $P_L(\beta, \alpha) = P_L(\gamma, \alpha)$ is non-trivial (Def. 5), the length of the loop must be 4 or more, i.e., each loop has at least 4 edges and 4 vertices, i.e.,

$$m_{i-1} + 2 \leq m_i \text{ for } 3 \leq i \leq l \quad (11.43)$$

The inequality (11.38) implies $m_2 \geq 2$. Together with $m_l = (s+t)/2$ and (11.43), we obtain

$$l \leq (s+t)/4 + 1, \quad t \geq 1. \quad (11.44)$$

Without loss of generality, we assume $m_i - m_{i-1}$ is creasing with $i \geq 3$, i.e., for $3 \leq i < j \leq l$

$$m_i - m_{i-1} \leq m_j - m_{j-1} \quad (11.45)$$

Denote by $N(\alpha, s, \{v_1, \dots, v_t\}, l, \{m_1, \dots, m_l\})$ the set of all pairs (β, γ) having the graph above and we now estimate the number of elements of this set.

Using the notions $W_i = (w_{2i-1}, w_{2i})$ and $\tilde{W}_i = (\tilde{w}_{2i-1}, \tilde{w}_{2i})$, we can add

the information between k_i 's and p_i 's into the graph as follows

$$\begin{aligned}
& k_1 \xleftrightarrow{W_1} p_1 \xleftrightarrow{\widetilde{W}_1} p_2 \xleftrightarrow{W_2} p_3 \cdots p_{2m_1-1} \xleftrightarrow{W_{m_1}} k_4 \text{ (or } k_2) \\
& k_3 \xleftrightarrow{\widetilde{W}_{m_1}} p_{2m_1} \xleftrightarrow{W_{m_1+1}} p_{2m_1+1} \cdots p_{2m_2-2} \xleftrightarrow{\widetilde{W}_{m_2}} k_2 \text{ (or } k_4) \\
& p_{2m_2-1} \xleftrightarrow{W_{m_2+1}} p_{2m_2} \xleftrightarrow{\widetilde{W}_{m_2+1}} p_{2m_2+1} \cdots p_{2(m_3)-2} \xleftrightarrow{\widetilde{W}_{m_3}} p_{2m_2-1} \\
& \dots \\
& \dots \\
& p_{2m_{l-1}-1} \xleftrightarrow{W_{m_{l-1}+1}} p_{2m_{l-1}} \xleftrightarrow{\widetilde{W}_{m_{l-1}+1}} p_{2m_{l-1}+1} \cdots p_{2(m_l)-2} \xleftrightarrow{\widetilde{W}_{m_l}} p_{2m_{l-1}-1} ,
\end{aligned} \tag{11.46}$$

where w_i 's are the union of s zero's and $\{v_1, \dots, v_t\}$, so are \widetilde{w} 's. More specifically, if $A \xleftrightarrow{W} B$ appears in the graph and $W = (C, D)$, then the operator $\mathcal{A}_{A,B}^{C,D}$ appears in (11.37). So we have

$$A \xleftrightarrow{W_i} B \Leftrightarrow A + B = w_{2i-1} + w_{2i},$$

so as \widetilde{W} 's. With this relation, we can see that β and γ is uniquely determined by the structure of the graph, w_i 's, \widetilde{w}_i 's and one k_i or p_i for each loop or chain.

To bound $|N(\widetilde{\alpha}, s, \{v_1, \dots, v_t\}, l, \{m_1, \dots, m_l\})|$, we note that the sum of momentum (p_i 's) in each loop is zero. Thus we can count the number of graphs as follows.

1. choose the positions of zeros in β edges. The total number of choices is less than 2^{t+s} .
2. choose the positions of $v_1 \cdots v_t$ in β edges. The total number of choices is $t!$.
3. choose the positions of zeros in γ edges. The total number of choices is less than 2^{t+s} again.
4. choose the positions of $v_1 \cdots v_t$ in γ edges. We call a loop trivial if all the momenta associated with γ edges are zero. The number of trivial loops is at most $s/4$ since there are at least two γ edges per loop. Hence the number of non-trivial loops is at least $l - s/4$. Thus we only have to fix v in at most $t - (l - s/4)$ edges and the number of choices is at most $t^{t-l+s/4}$.

Thus we obtain

$$\begin{aligned}
& |N(\alpha, s, \{v_1, \dots, v_t\}, l, \{m_1, \dots, m_l\})| \\
& \leq (\text{const.})^{t+s} t! t^{(t+s/4-l)} (\varrho^{-3\eta} \Lambda)^l \\
& \leq (\text{const.})^{t+s} t! t^{(3t/4)} (\varrho^{-3\eta} \Lambda)^{t/4+s/4+1}
\end{aligned} \tag{11.47}$$

where we have used (11.44) Since

$$|N(\alpha, s, \{v_1, \dots, v_t\})| = \sum_l \sum_{\{m_1, \dots, m_l\}} |N(\alpha, s, \{v_1, \dots, v_t\}, l, \{m_1, \dots, m_l\})|$$

and

$$\sum_l \sum_{\{m_1, \dots, m_l\}} 1 \leq \text{const.}^{s+t} \tag{11.48}$$

we have proved (11.34). ■

12 Proofs of Lemmas 1, 2, 3

12.1 Proof of Lemma 1

The proof of Lemma 1 is standard and only a sketch will be given. We first construct an isometry between functions with periodic boundary condition in $\Lambda = [0, L]^3$ and functions with Dirichlet boundary condition in $\Lambda^* = [-\ell, L + \ell]^3$, where $L = \varrho^{-41/60}$ and $\ell = \varrho^{-41/120}$. We note, by the definition of ϱ^* in (3.5),

$$|\Lambda| \varrho = |\Lambda^*| \varrho^* \tag{12.1}$$

Denote the coordinates of \mathbf{x} by $\mathbf{x} = (x^{(1)}, x^{(2)}, x^{(3)})$. Let $h(\mathbf{x})$ supported on $[-\ell, L + \ell]^3$ be the function $h(\mathbf{x}) = q(x^{(1)})q(x^{(2)})q(x^{(3)})$ where

$$q(x) = \begin{cases} \cos[(x - \ell)\pi/4\ell], & |x| \leq \ell \\ 1, & \ell < x < L - \ell \\ \cos[(x - (L - \ell))\pi/4\ell], & |x - L| \leq \ell \\ 0, & \text{otherwise} \end{cases} \tag{12.2}$$

The function $q(x)$ is symmetric w.r.t $x = L/2$. Due to the property of cosine, for any function ϕ with the period L we have

$$\int_{\mathbf{x} \in [-\ell, L + \ell]^3} |h\phi(\mathbf{x})|^2 d\mathbf{x} = \int_{\mathbf{x} \in [0, L]^3} |\phi(\mathbf{x})|^2 d\mathbf{x} \tag{12.3}$$

Thus the map $\phi \longrightarrow h\phi$ is an isometry:

$$L^2_{\text{Periodic}}([0, L]^3) \rightarrow L^2_{\text{Dirichlet}}([-\ell, L + \ell]^3).$$

Let $\chi(\mathbf{x})$ be the characteristic function of the ℓ -boundary of $[-\ell, L + \ell]^3$, i.e., $\chi(\mathbf{x}) = 1$ if $|x^{(\alpha)}| \leq \ell$ for some $\alpha = 1, 2$ or 3 where $|x^{(\alpha)}|$ is the distance on the torus. Then standard methods yield the following estimate on the kinetic energy of $h\phi$

$$\begin{aligned} & \int_{\mathbf{x} \in [-\ell, L + \ell]^3} |\nabla(h\phi)(\mathbf{x})|^2 \\ & \leq \int_{\mathbf{x} \in [0, L]^3} |\nabla\phi(\mathbf{x})|^2 + \text{const.} \ell^{-2} \int \chi(\mathbf{x}) |\phi(\mathbf{x})|^2 \end{aligned} \quad (12.4)$$

The generalization of this isometry to higher dimensions is straightforward. Suppose $\Psi(\mathbf{x}_1, \dots, \mathbf{x}_N) \in L^2_{\text{periodic}}([0, L]^{3N})$. Here

$$N = |\Lambda| \varrho = |\Lambda^*| \varrho^* \quad (12.5)$$

Then for any $u \in \mathbb{R}^3$, the map

$$\mathcal{F}^u(\Psi) := \Psi(\mathbf{x}_1, \dots, \mathbf{x}_i, \dots, \mathbf{x}_N) \prod_{i=1}^N h(x_i + u) h(y_i + u) h(z_i + u) \quad (12.6)$$

is an isometry from $L^2_{\text{Periodic}}([0, L]^{3N})$ to $L^2_{\text{Dirichlet}}([-\ell - u, L + \ell - u]^{3N})$. Here the \mathbf{x} 's in Ψ are coordinates on the torus. Clearly, \mathcal{F}^u has the property (12.4).

The potential V can be extended to be periodic by defining $V^P(x - y) = V([x - y]_P)$ where $[x - y]_P$ is the difference of x and y as elements on the torus $[0, L]$. Since V is nonnegative and has fast decay in the position space, we have $V(x - y) \leq V^P(x - y)$. From the definition of \mathcal{F}^u , we conclude that

$$\int_{[-\ell - u, L + \ell - u]^{3N}} |\mathcal{F}^u(\Psi)|^2 V(\mathbf{x}_1 - \mathbf{x}_2) \prod_{i=1}^N dx_i \leq \int_{[0, L]^{3N}} |\Psi|^2 V^P(\mathbf{x}_1 - \mathbf{x}_2) \prod_{i=1}^N dx_i$$

Therefore, the energy of two boundary conditions are related by

$$\langle H_N \rangle_{\mathcal{F}^u(\Psi)} \leq \langle H_N \rangle_{\Psi} + \text{const.} \ell^{-2} \sum_{i=1}^N \langle \chi(\mathbf{x}_i + u) \rangle_{\Psi} \quad (12.7)$$

We note \mathcal{F}^u is operator on pure state Ψ . It can be generalized to operator \mathcal{G}^u on state Γ as follows. For any state Γ^P of N particles in $[0, L]^3$ with periodic boundary condition, we define

$$\mathcal{G}^u(\Gamma^P) := \mathcal{F}^u \Gamma^P (\mathcal{F}^u)^\dagger \quad (12.8)$$

So $\Gamma^D = \mathcal{G}^u(\Gamma^P)$ is a state of N particles in $[-\ell - u, L + \ell - u]^3$ with Dirichlet boundary condition. With (12.1), one can see

$$\mathcal{G}^u : \Gamma^P(\varrho, \Lambda, \beta) \rightarrow \Gamma^D(\varrho^*, \Lambda^*, \beta) \quad (12.9)$$

Using (12.7), we have:

$$\text{Tr } H_N \mathcal{G}^u(\Gamma^P) \leq \text{Tr } H_N \Gamma^P + \text{const. } \ell^{-2} \sum_{i=1}^N \text{Tr} \chi(x_i + u) \Gamma^P \quad (12.10)$$

Averaging over $u \in [0, L]^3$, we have

$$L^{-3} \int (\text{Tr } H_N \mathcal{G}^u(\Gamma^P)) du \leq \text{Tr } H_N \Gamma^P + \text{const. } \ell^{-1} L^{-1} N \quad (12.11)$$

So for any Γ^P there exists at least one u such that

$$\text{Tr } H_N \mathcal{G}^u(\Gamma^P) \leq \text{Tr } H_N \Gamma^P + \text{const. } N \left(\frac{1}{\ell L} \right) \quad (12.12)$$

On the other hand, the fact \mathcal{F}^u ((12.6)) is a isometry implies that $\mathcal{G}^u(\Gamma^P)$ and Γ^P have the same von-Neumann entropy, i.e.,

$$S(\mathcal{G}^u(\Gamma^P)) = S(\Gamma^P) \quad (12.13)$$

Combine (12.12) and (12.13), we obtain Δf the free energy difference between $\mathcal{G}^u(\Gamma^P)$ and Γ^P is less than $\text{const. } N \left(\frac{1}{\ell L} \right)$. With the choice $L = \varrho^{-41/60}$ and $\ell = \varrho^{-41/120}$, the error term is negligible to the accuracy we need in proving Lemma 1. This concludes the proof of Lemma 1.

12.2 Proof of Lemma 2

It is not easy to construct Γ_0 (the state of N particles) directly. So we start with constructing a state $\Gamma_{\mathcal{F}}$ in Fock space. Then pick up the useful component of $\Gamma_{\mathcal{F}}$ and revise it to Γ_0 .

First, let $B_{\mathcal{F}}$ be the standard basis of the Fock space $\mathcal{F}(\Lambda)$ as follows

$$B_{\mathcal{F}} \equiv \left\{ |\alpha\rangle : |\alpha\rangle = C_{\alpha} \prod_{k \in (\frac{2\pi\mathbb{Z}}{L})^3} (a_k^{\dagger})^{\alpha(k)} |0\rangle, \alpha(k) \in \mathbb{N} \cup \{0\} \right\}, \quad (12.14)$$

where C_{α} is a positive normalization constant. We define a new Hamiltonian \tilde{H} of free particles in $[0, L]^3$ with the following revised Bose statistics, i.e.,

1. The number of the particles in single particle state $|k\rangle$ is nonzero only when $k \in P_I \cup P_L$.
2. The number of the particles in single particle state $|k\rangle$, $k \in P_L \cup P_I$, must be no more than C_k , which will be chosen later.

With the definition of μ in (2.8), we define $\Gamma_{\mathcal{F}}$ as the grand-canonical Gibbs state of \tilde{H} with the chemical potential $\mu(\tilde{\varrho}, \beta) \leq 0$ and temperature β^{-1} , where

$$\tilde{\varrho} \equiv \varrho(1 - L^{-1/2}) = \varrho(1 - o(\varrho^{1/3})) \quad (12.15)$$

and C_k is chosen as follows (Recall $m_c = \varrho^{-3\eta}$)

$$C_k = \begin{cases} \frac{(m_c)^{1/3}}{\beta E_{k,\mu}} & k \in P_I \\ m_c & k \in P_L \end{cases}, \quad (12.16)$$

where $E_{k,\mu}$ is defined as $k^2 - \mu(\tilde{\varrho}, \beta)$. We note that $\beta \sim \varrho^{-2/3}$ implies $\beta E_{k,\mu} C_k \gg |\log \varrho|$.

We claim that the state $\Gamma_{\mathcal{F}}$ has the following properties:

Lemma 20. *The free energy per volume of $\Gamma_{\mathcal{F}}$ is bounded above by*

$$f(\Gamma_{\mathcal{F}}) \leq f_0(\varrho, \beta)(1 - o(\varrho^{1/3})) \quad (12.17)$$

In most cases, the total particle number of $\Gamma_{\mathcal{F}}$ is less than $N = \varrho\Lambda$, i.e.,

$$\sum_{m=1}^N \text{Tr}_{\mathcal{H}_m} \Gamma_{\mathcal{F}}^m \geq 1 - \varrho \quad (12.18)$$

Here $\Gamma_{\mathcal{F}}^m$ are the components of $\Gamma_{\mathcal{F}}$ on \mathcal{H}_m , i.e.,

$$\Gamma_{\mathcal{F}} = \sum_{m=0}^{\infty} \Gamma_{\mathcal{F}}^m, \quad \Gamma_{\mathcal{F}}^m : \mathcal{H}_m \rightarrow \mathcal{H}_m \quad (12.19)$$

Similarly, in most cases, the total particle number of $\Gamma_{\mathcal{F}}$ is very close to $\min\{\varrho, \varrho_c\}\Lambda$. We have

$$\sum_{|m - \min\{\varrho, \varrho_c\}\Lambda| \leq N\varrho^{1/3}} \text{Tr}_{\mathcal{H}_m} \Gamma_{\mathcal{F}}^m \geq 1 - \varrho \quad (12.20)$$

Proof. proof of Lem.20

First, we prove (12.17), by the definition, the free energy of $\Gamma_{\mathcal{F}}$ is

$$\begin{aligned} & \frac{-1}{\beta} \left[\sum_{k \in P_L \cup P_I} \log \left(\frac{e^{\beta E_{k,\mu}} - e^{-\beta E_{k,\mu} C_k}}{e^{\beta E_{k,\mu}} - 1} \right) \right] \\ & + \sum_{k \in P_L \cup P_I} \mu(\tilde{\varrho}, \beta) \left(\frac{1}{e^{\beta E_{k,\mu}} - 1} - \sum_{k \in P_L \cup P_I} \frac{1 + C_k}{e^{\beta E_{k,\mu}(C_k+1)} - 1} \right), \end{aligned} \quad (12.21)$$

With the definition of P_I and P_L , one can remove the C_k terms, add the $k \notin P_I \cup P_L$ terms and check that (12.21) is equal to

$$\left(\frac{-1}{\beta} \sum_{k \in (\frac{2\pi\mathbb{Z}}{L})^3, k \neq 0} \log \left(\frac{e^{\beta E_{k,\mu}}}{e^{\beta E_{k,\mu}} - 1} \right) + \sum_{k \in (\frac{2\pi\mathbb{Z}}{L})^3, k \neq 0} \mu(\tilde{\varrho}, \beta) \frac{1}{e^{\beta E_{k,\mu}} - 1} \right) (1 + o(\varrho^{1/3})), \quad (12.22)$$

Then with the choice $L = \varrho^{-41/60}$ and the definition of free energy f_0 in (2.6) and (2.7), we have

$$(12.22) = f_0(\tilde{\varrho}, \beta) \Lambda (1 + o(\varrho^{1/3})) \quad (12.23)$$

Combine with $\tilde{\varrho} = \varrho(1 + o(\varrho^{1/3}))$, we obtain the desired result (12.17).

Then we prove (12.18). Let $n(k)$ denote the number of the particles in one-particle-state $|k\rangle$. So $\overline{n(k)}$ the average of $n(k)$ is equal to $\text{Tr} a_k^\dagger a_k \Gamma_{\mathcal{F}}$. By the definition, the average total number of particles of $\Gamma_{\mathcal{F}}$ is equal to

$$\sum_{k \in P_I \cup P_L} \overline{n(k)} = \sum_{k \in P_I \cup P_L} \frac{1}{e^{\beta E_{k,\mu}} - 1} - \sum_{k \in P_L \cup P_I} \frac{1 + C_k}{e^{\beta E_{k,\mu}(C_k+1)} - 1} \quad (12.24)$$

Similarly, with $L = \varrho^{-41/60}$ and $\beta E_{k,\mu} C_k \gg |\log \varrho|$, one can easily prove:

$$(12.24) = \min\{\tilde{\varrho}, \varrho_c\} \Lambda (1 + O(\varrho^{-1/3} L^{-1} \log \varrho)) \quad (12.25)$$

On the other hand, since $n(k)$ are independent random variables and they are bounded in (12.16), we can use Hoeffding's inequality [6] to estimate

the distribution of the total particle number of $\Gamma_{\mathcal{F}}$. With $n(k) \leq C_k$ and Hoeffding's inequality [6], we obtain that the probability of find more than N particles in $\Gamma_{\mathcal{F}}$ is bounded above by

$$P\left(\sum_k n(k) > N\right) \leq \exp\left\{-\frac{\left[N - \sum_k \overline{n(k)}\right]^2}{\sum_{k \in P_I \cup P_L} C_k^2}\right\} \quad (12.26)$$

By the definition of C_k (12.16), we have:

$$\sum_{k \in P_I \cup P_L} C_k^2 = O(\varrho^{4/3} \Lambda L m_c^{2/3}) \quad (12.27)$$

On the other hand, with the fact $\varrho - \tilde{\varrho} = \varrho L^{-1/2}$ and (12.25), we can see that

$$\left[N - \sum_k \overline{n(k)}\right]^2 \geq O(\varrho^2 L^5) \quad (12.28)$$

Inserting $L = \varrho^{-41/60}$, (12.27) and (12.28) into (12.26), we obtain the desired result (12.18). And (12.20) can be proved similarly with (12.25) and (12.27). \blacksquare

By Lemma 20, there exists m_0 s.t.

$$m_0 \leq N \text{ and } |m_0 - \min\{\varrho, \varrho_c\} \Lambda| \leq \varrho^{1/3} N \quad (12.29)$$

and the free energy of $\Gamma_{\mathcal{F}}^{m_0}$ is less than $f_0(\varrho, \beta) \Lambda (1 - o(\varrho^{1/3}))$.

Then adding $N - m_0$ ($N = \varrho \Lambda$) particles with momentum zero into the system described by $\Gamma_{\mathcal{F}}^{m_0}$, we obtain a new state Γ_0 of N free particles. The state Γ_0 always has $N - m_0$ particles with momentum zero and the free energy of Γ_0 is also less than $f_0(\varrho, \beta) \Lambda (1 - o(\varrho^{1/3}))$, i.e.,

$$\left|\text{Tr} - \Delta \Gamma_0 + \frac{1}{\beta} S(\Gamma_0) - f_0(\varrho, \beta)\right| \Lambda^{-1} \leq o(\varrho^2) \quad (12.30)$$

Furthermore, by the definition of $\Gamma_{\mathcal{F}}$, Γ_0 has the form:

$$\Gamma_0 = \sum_{\alpha \in M} g_{\alpha}(\varrho, \beta) |\alpha\rangle \langle \alpha|, \quad \alpha(0) = N - m_0 \text{ and } \sum_{\alpha \in M} g_{\alpha} = 1 \quad (12.31)$$

We note: if $\alpha(k) > C_k$ for some $k \in P_I \cup P_L$, then $g_{\alpha}(\varrho, \beta) = 0$. This property implies that the total number of the particles with momentum in P_I is $o(N)$, i.e., for any $\alpha \in M$

$$\sum_{\alpha \in M} \sum_{k \in P_I} g_{\alpha}(\varrho, \beta) \alpha(k) \ll N, \quad \alpha \in M \quad (12.32)$$

Together with (12.29) and the fact $\alpha(k) \leq m_c$ for $\alpha \in P_L$, we obtain (3.23).

At last we prove (3.22). First with the structure of Γ_0 , we have

$$\begin{aligned} \text{Tr}_{\mathcal{H}_{N,\Lambda}} V \Gamma_0 &= \sum_{\alpha \in M} g_\alpha(\varrho, \beta) \langle \alpha | V | \alpha \rangle \\ &= \sum_{\alpha \in M} g_\alpha(\varrho, \beta) \left(\sum_{k \in P_0 \cup P_I \cup P_L} V_0 \Lambda^{-1} (\alpha(k)^2 - \alpha(k)) \right. \\ &\quad \left. + \sum_{\substack{k \neq k' \\ k, k' \in P_0 \cup P_I \cup P_L}} 2(V_0 + V_{k-k'}) \Lambda^{-1} \alpha(k) \alpha(k') \right) \end{aligned} \quad (12.33)$$

Using the smoothness of V and $|k|, |k'| \ll 1$, we can replace $V_{k-k'}$ with V_0 without changing the leading term. Then with the cutoff C_k 's, the fact $\alpha(0) = N - m$ and (12.29), we have

$$\lim_{\varrho \rightarrow 0} |\text{Tr} V \Gamma_0| \varrho^{-2} \Lambda^{-1} = V_0 (2 - [1 - R[\beta]]_+^2) \quad (12.34)$$

Combine with (12.30), we obtain (3.22).

12.3 Proof of Lemma 3

Proof. Since the states $|\alpha\rangle$'s in M are orthonormal, we can rewrite the entropy of Γ_0 in lemma 2 as

$$S(\Gamma_0) = - \sum_{\alpha \in M} g_\alpha \log g_\alpha \quad (12.35)$$

For $S(\Gamma)$, we define A_∞ as $A_\infty \equiv \|\sum_{\alpha \in M} |\Psi_\alpha\rangle \langle \Psi_\alpha|\|_\infty$ and rewrite Γ as

$$\Gamma = A_\infty \sum_{\alpha \in M} g_\alpha \frac{|\Psi_\alpha\rangle \langle \Psi_\alpha|}{\sqrt{A_\infty}} \frac{\langle \Psi_\alpha|}{\sqrt{A_\infty}} \quad (12.36)$$

With the fact $\text{Tr} \Gamma = 1$, i.e., $\sum g_\alpha = 1$, we have

$$S(\Gamma) = -\log A_\infty - A_\infty \text{Tr} \left[\sum_{\alpha \in M} g_\alpha \frac{|\Psi_\alpha\rangle \langle \Psi_\alpha|}{\sqrt{A_\infty}} \frac{\langle \Psi_\alpha|}{\sqrt{A_\infty}} \log \left(\sum_{\alpha \in M} g_\alpha \frac{|\Psi_\alpha\rangle \langle \Psi_\alpha|}{\sqrt{A_\infty}} \frac{\langle \Psi_\alpha|}{\sqrt{A_\infty}} \right) \right] \quad (12.37)$$

With Berezin-Lieb inequality [11], [2], we obtain

$$S(\Gamma) \geq -\log A_\infty - \sum_{\alpha \in M} g_\alpha \log g_\alpha = -\log A_\infty + S(\Gamma_0) \quad (12.38)$$

On the other hand, we claim the following lemma

Lemma 21.

$$\lim_{\varrho \rightarrow 0} \left(\log \left\| \sum_{\alpha \in M} |\Psi_\alpha\rangle \langle \Psi_\alpha| \right\|_\infty \right) \frac{1}{N \varrho^{1/3}} = 0 \quad (12.39)$$

Insert this lemma into (12.38), we arrive at the desired result (3.27). \blacksquare

12.3.1 Proof of Lemma 21

Proof. With the fact: for any hermitian matrix $M = M_{ij}$,

$$\|M\|_\infty \leq \max_i \left\{ \sum_j |M_{ij}| \right\},$$

we can bound $\|\sum_{\alpha \in M} |\Psi_\alpha\rangle \langle \Psi_\alpha|\|_\infty$ as follows (Recall \widetilde{M} in Def. 2.)

$$\begin{aligned} \left\| \sum_{\alpha \in M} |\Psi_\alpha\rangle \langle \Psi_\alpha| \right\|_\infty &\leq \max_{\beta \in \widetilde{M}} \left\{ \sum_{\alpha \in M} \sum_{\gamma \in \widetilde{M}} |\langle \beta | \Psi_\alpha \rangle \langle \Psi_\alpha | \gamma \rangle| \right\} \\ &\leq \max_{\beta \in \widetilde{M}} \left\{ \sum_{\alpha \in M} |\langle \beta | \Psi_\alpha \rangle| \right\} \cdot \max_{\alpha \in M} \left\{ \sum_{\gamma \in \widetilde{M}} |\langle \gamma | \Psi_\alpha \rangle| \right\}, \end{aligned} \quad (12.40)$$

With the fact Ψ_α is the linear combination of states in $M_\alpha \subset \widetilde{M}_\alpha$ and $|\beta\rangle, |\Psi_\alpha\rangle$ are normalized, we claim

$$\log \left(\max_{\beta \in \widetilde{M}} \left\{ \sum_{\alpha \in M} |\langle \beta | \Psi_\alpha \rangle| \right\} \right) \leq \varrho^{1-4\eta-3\kappa_L} \quad (12.41)$$

$$\log \left(\max_{\alpha \in M} \left\{ \sum_{\gamma \in \widetilde{M}} |\langle \gamma | \Psi_\alpha \rangle| \right\} \right) \leq \varrho^{1-4\eta-3\kappa_L} + \varrho^{-4\eta-3\kappa_H} \quad (12.42)$$

First, we prove (12.41). We know $|\langle \beta | \Psi_\alpha \rangle| \neq 0$ implies $|\langle \beta | \Psi_\alpha \rangle| \leq 1$, $\alpha \in M$ and $\beta \in \widetilde{M}_\alpha$. With the definition of M and \widetilde{M}_α , if $\alpha \in M$, $\beta \in \widetilde{M}_\alpha$, then

$$\begin{aligned} \beta(u) &= \alpha(u) \text{ for } u \in P_I \\ \beta(u) &\leq \alpha(u) \text{ for } u \in P_L \\ \alpha(u) &= 0 \text{ for } u \in P_H \end{aligned} \quad (12.43)$$

and for any fixed small box $B_L^i (i = 1, 2, \dots)$ in P_L , $\beta(u)$ is very close to $\alpha(u)$, s.t.

$$\sum_{u \in B_L^i} |\beta(u) - \alpha(u)| \leq 1 \quad (12.44)$$

Now let's count, for fixed β , how many $\alpha \in M$ satisfying $\beta \in \widetilde{M}_\alpha$. This number must be less than the α 's satisfying (12.43) and (12.44). By the definition of B_L 's, the total number of B_L 's is less than $\text{const. } \varrho^{1-3\eta-3\kappa_L}$. And for any B_L^i , $|B_L^i|$ the number of the elements in B_L^i is less than $\text{const. } \varrho^{3\kappa_L} \Lambda$. Therefore, for fix $\beta \in \widetilde{M}$, the total number of $\alpha \in M$ satisfying $\beta \in \widetilde{M}_\alpha$ is less than

$$(\text{const. } \varrho^{3\kappa_L} \Lambda)^{\text{const. } \varrho^{1-3\eta-3\kappa_L}} \quad (12.45)$$

Together with the fact $|\langle \beta | \Psi_\alpha \rangle| \leq 1$, we proved (12.41).

Then we prove (12.42). Similarly, using the rule 2 of Def. 3, we can count, for fix $\alpha \in M$, the total number of $\gamma \in \widetilde{M}$, s.t. $|\langle \gamma | \Psi_\alpha \rangle| \neq 0$ is less than

$$(\text{const. } \varrho^{3\kappa_L} \Lambda)^{\text{const. } \varrho^{1-3\eta-3\kappa_L}} (\text{const. } \varrho^{3\kappa_H} \Lambda)^{\text{const. } \varrho^{-3\eta-3\kappa_H}}, \quad (12.46)$$

which implies (12.42). Inserting (12.41) and (12.42) into (12.40), we obtain the desired result (12.39). \blacksquare

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